# Precalculus Review Examples: Trigonometry

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These examples cover the high-school level technical skills needed to succeed in the study of calculus.

To help emphasize the techniques, we'll be very explicit in these notes with comments on each step in each solution, indicating why we chose to take a certain action.

When working problems yourself, we encourage you to be more explicit and careful in writing up solutions, even to simple problems. Many difficulties are solved just by showing more steps and organizing your work.

## Contents

# 1 Trigonometry

Here are some issues and strategies to remember when solving equations with trigonometry.

- Often there are many ways to approach a problem. Using different identities may lead to the same correct solution.
- The domain issues on inverting trig functions are tricky, since the functions are periodic. You have to know what kind of numbers you are looking for.
- Often reducing to sin and cos, and even further to just one of the two when possible, is a very useful strategy.
- The use of trig identities is a skill that comes with practice. It's often a case of building up an intuition of what will work for a give equation.
- Use the formula sheet, and try to match types. If the equation involves squares, look for identities with squares. If the equation involves sums, look for identities with sums.

Here are some examples of equations with trigonometry in one variable. In each case, we are solving for the unknown variable.

## 1.1 Trigonometric Equations

### 1.1.1 Example

$$\cos\left(x + \frac{\pi}{3}\right) = \frac{-1}{\sqrt{2}}$$

Since there is an isolated cosine, we can apply the inverse cosine to solve the equation.

$$\begin{aligned} \arccos\cos\left(x + \frac{\pi}{3}\right) &= \arccos\frac{-1}{\sqrt{2}} \\ x + \frac{\pi}{3} &= \arccos\frac{-1}{\sqrt{2}} \end{aligned}$$

Calculating inverse cosine for given values can be done with a calculator, or in this case, referencing certain well known triangles. The special trianges are listed in the reference material. In addition, it is useful to remember the sign of sine and cosine in each of the four quadrants.

$$\begin{aligned} x + \frac{\pi}{3} &= \frac{5\pi}{4} \\ x &= \frac{5\pi}{4} - \frac{\pi}{3} = \frac{11\pi}{12} \end{aligned}$$

### 1.1.2 Example

$$\sin x + \cos^2 x = 2$$

It's difficult to work with both sine and cosing in the same equation. We'd like to change the form to one involving all sine term or all cosine terms. We have an identitiy for squares, so we can use that identity change $\cos^2 x$ into $1 - \sin^2 x$.

$$\begin{aligned} \sin x + 1 - \sin^2 x &= 2 \\ \sin x = \sin^2 x &= 1 \\ \sin^2 x - \sin x + 1 &= 0 \end{aligned}$$

This is a quadratic in $\sin x$, which we can try to solve.

$$\sin x = \frac{1 \pm \sqrt{1-4}}{2}$$

Since there is a negative square root, there are no solutions.

### 1.1.3 Example

$$\tan^2 x + \sec^2 x = 1$$

With various trig functions, its often useful to change everything back to sine and cosine to see if we can rearrange the equation into something useful.

$$\frac{\sin^2 x}{\cos^2 x} + \frac{1}{\cos^2 x} = 1$$

We can multiply both sides by $\cos^2 x$.

$$\sin^2 x + 1 = \cos^2 x$$

Now we can converse the $\sin^2 x$ to $1 - \cos^2 x$ using the square identity.

$$\begin{aligned} 1 - \cos^2 x + 1 &= \cos^2 x \\ 2 &= 2\cos^2 x \\ \cos^2 x &= 1 \\ \cos x &= \pm 1 \\ x &= \arccos(\pm 1) \end{aligned}$$

With inverting trig, we always have the possibility of many solutions. The standard inverses would give $\arccos(1) = 0$ and $\arccos(-1) = \pi$ but any other integer multiple of $\pi$ would also give $\pm 1$ under cosine.

### 1.1.4 Example

$$\cos 2x - \sin^2 x = 0$$

$\cos 2x$ has an identity as $\cos^2 x - \sin^2 x$.

$$\begin{aligned} \cos^2 x - \sin^2 x - \sin^2 x &= 0 \\ \cos^2 x - 2\sin^2 x &= 0 \end{aligned}$$

$\cos^2 x = 1 - \sin^2 x$ is a good replacement to change everything into sine.

$$\begin{aligned} 1 - \sin^2 x - 2\sin^2 x &= 0 \\ 1 &= 3\sin^2 x \\ \frac{1}{3} &= \sin^2 x \\ \pm\sqrt{\frac{1}{3}} &= \sin x \\ \arcsin \pm\sqrt{\frac{1}{3}} &= x \end{aligned}$$

Again, there may be many possible solutions since sine is periodic. There are no special triangles where sin is 1/3, so we'd have to ask a computer for an approximate answer.

## 1.2 Other Trigonometric Problems

### 1.2.1 Example

$$\cos(3+x) - \sin(3-x) = 0$$

This is an example which uses the addition identities. Those identities give:

$$\begin{aligned} \cos 3 \cos x - \sin 3 \sin x - [\sin 3 \cos x - \cos 3 \sin x] &= 0 \\ \cos 3 \cos x - \sin 3 \sin x - \sin 3 \cos x + \cos 3 \sin x &= 0 \end{aligned}$$

It would be nice to factor this – can we see a way? Each term with cos is positive, and each with $\sin 3$ is negative. That might lead to the following idea

$$(\cos 3 - \sin 3)(\cos x + \sin x) = 0$$

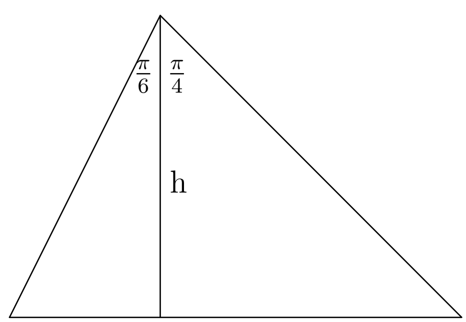
We can divide by the constant term to get

$$\begin{aligned} \cos x + \sin x &= 0 \\ \sin x &= -\cos x \\ \tan x &= -1 \\ x &= \arctan(-1) \\ x &= \frac{-\pi}{4}, \quad \frac{3\pi}{4}, \quad \frac{7\pi}{4} \quad \ldots \end{aligned}$$

Here are some more geometrically motivated trig examples.

### 1.2.2 Example

Consider the following triangle.




Find an expression for the area of the large in terms of the height $h$. Manipulate that expression to express $h$ in terms of $A$. Find which height leads to an area of $4m^2$.

To solve this, we calculate the areas of the two smaller triangles. We need the lengths of their bases, which we can call $d_1$ and $d_2$. In each case, for the angle at the top of the triangle $\theta_1$ or $\theta_2$, the appropriate trig ration is $\tan\theta_1 = \frac{d_1}{h}$ and $\tan\theta_2 = \frac{d_2}{h}$. These re-arrange as

$$d_1 = h\tan\theta_1 = h\frac{\sqrt{3}}{3}$$

$$d_2 = h\tan\theta_2 = h \cdot 1 = h$$

Then the area of a triangle is base times height over 2, so the total area is

$$A = \frac{hd_1}{2} + \frac{hd_2}{2} = \frac{h^2\sqrt{3}}{6} + \frac{h^2}{2} = h^2\left(\frac{\sqrt{3}}{6} + \frac{1}{2}\right) = h^2\left(\frac{\sqrt{3}+3}{6}\right)$$

Now we try to isolate h.

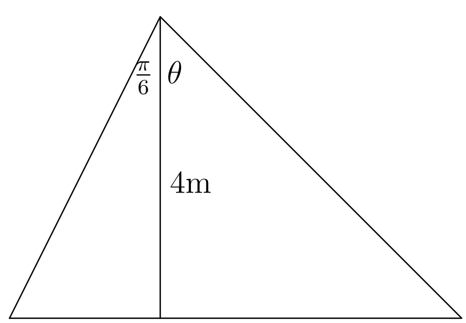
$$h^2 = \frac{6A}{\sqrt{3}+3}$$
$$h = \sqrt{\frac{6A}{\sqrt{3}+3}}$$

Now we insert $A = 4$ to answer the last part. This calculation can be done with a calculator.

$$h \doteq 2.25m$$

### 1.2.3 Example

Now consider a similar triangle, but with a fixed height and one angle variable:




Find an expression for the area A, solve that to express $\theta$ in terms of $A$, and calculate which $\theta$ gives an area of $15m^2$.

The area calculation is similar to above:

$$A = \frac{hd_1}{2} + \frac{hd_2}{2} = \frac{h^2 \tan\frac{\pi}{6}}{2} + \frac{h^2 \tan\theta}{2} = \frac{4^2\sqrt{3}}{6} = \frac{4^2\sqrt{3}}{6} + \frac{4^2 \tan\theta}{2} = \frac{8\sqrt{3}}{3} + 8\tan\theta$$

Now we try to isolate $\theta$.

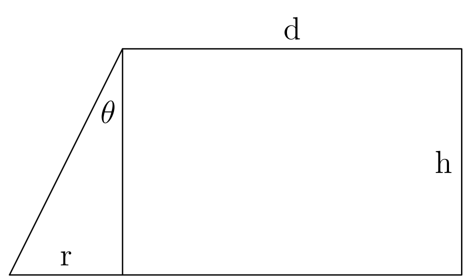
$$8\tan\theta = A - \frac{8\sqrt{3}}{3}$$
$$\tan\theta = \frac{A}{8} - \frac{\sqrt{3}}{3}$$
$$\theta = \arctan\left(\frac{A}{8} - \frac{\sqrt{3}}{3}\right)$$

Now we input $A = 15$ to solve for the specific value. A calculator can be used here.

$$\theta = \arctan\left(\frac{15}{8} - \frac{\sqrt{3}}{3}\right) \doteq 0.91\text{rad} \doteq 52.4^\circ$$

#### 1.2.4 Example

Finally, here's another interesting geometric problem. Consider the following trapezoid:




Find an expression for the area of this trapezoid in terms of $d$, $h$ and $\theta$. Then given $\theta = \frac{\pi}{6}$ and $d = 10m$, calculate which $h$ gives a total area of $30m^2$.

We can use the material from the previous section. The height of the triangle on the left side is $h$, and its top angle is $\theta$, so as calculated in the previous sections $r = h\tan\theta$. Then the area of the triangle is $h^2\tan\theta/2$. With the area of the rectangle of $dh$, the total area is

$$A = dh + \frac{h^2 \tan\theta}{2}$$

Inserting the values given

$$\begin{aligned}
30 &= 10h + \frac{h^2 \tan \frac{\pi}{6}}{2} \\
30 &= 10h + \frac{h^2\sqrt{3}}{6} \\
0 &= \frac{h^2\sqrt{3}}{6} + 10h - 30 \\
0 &= \sqrt{3}h^2 + 60h - 180
\end{aligned}$$

This is a quadratic in $h$, which we can solve with the quadratic formula

$$\begin{aligned}
h &= \frac{-60 \pm \sqrt{60^2 - 4(\sqrt{3})(-180)}}{2\sqrt{3}} \\
h &= \frac{-60 \pm \sqrt{3600 + 720\sqrt{3}}}{2\sqrt{3}}
\end{aligned}$$

The negative choice for $\pm$ gives a negative height which is meaningless, so the positive choice must be correct

$$h = \frac{-60 + \sqrt{3600 + 720\sqrt{3}}}{2\sqrt{3}}$$

A calculator can be used to calculate this

$$h \doteq 2.78$$

This is a good time to mention the utility of checking your answers against logic. When I originally did this calculation, I came up with a height of $6.5m$. That may seem like the right order of magnitude, but at that height, the rectangle alone has area $65m^2$, certainly larger than the desired $30m^2$, so I knew I had to go back and find the error in my calculation.