Precalculus Review Examples: Exponents and Logarithms

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September 2016

These examples cover the high-school level technical skills needed to succeed in the study of calculus.

To help emphasize the techniques, we'll be very explicit in these notes with comments on each step in each solution, indicating why we chose to take a certain action.

When working problems yourself, we encourage you to be more explicit and careful in writing up solutions, even to simple problems. Many difficulties are solved just by showing more steps and organizing your work.

Contents

1 Exponents and Logarithms

The key idea here is to use the laws of exponents and logarithms to put the equations in reasonable form and use the fact that exponents and logarithms cancel each other out to isolate the variable.

1.1 What is a Logarithm?

As an example, let's start with this:

$$10^3 = 1000.$$

Then the definition of the "logarithm base 2" is

$$\log_{10} 1000 = 3.$$

The $\log_{10}$ function asks the question, "What power of 2 will give me 8?" We can see this more clearly by rewriting it as

$$\log_{10} 10^3 = 3.$$

The same definition works in any base:

$$\log_2 2^5 = 5 \qquad \log_3 3^{38} = 38 \qquad \log_5 5^{1.7} = 1.7 \quad \text{etc.}$$

We can use this to solve equations where the variable is part of an exponent. Say we want to solve

$$2^x = 16.$$

The definition of logarithms tells us that $\log_2 2^x = x$, so we can "bring down" the x by taking the log of both sides:

$$\log_2 2^x = \log_2 16$$

The left side becomes x. On the right, $16 = 2^4$ so $\log_2 16 = 4$. Thus the solution is

$$x = 4.$$

1.2 Exponential and Logarithmic Equations

1.2.1 Example

$$\left(\frac{1}{2}\right)^{3x} = 64$$

With a fraction $\frac{1}{2}$, we might try a logarithm base 2 to simplify. Remember that we must apply any operation to both sides of the equation.

$$\log_2 \left(\frac{1}{2}\right)^{3x} = \log_2 64 = 6$$

We can use one of the logarithmic manipulation rules to take the exponent out of the logarithm.

$$\begin{aligned} 3x\log_2\frac{1}{2} &= 6 \\ 3x(-1) &= 6 \\ x &= -2 \end{aligned}$$

1.2.2 Example

$$3^{4x-1} = 27$$

We apply a logarithm base 3.

$$\log_3 3^{4x-1} = \log_3 27 = 3$$

We can then take out the exponent.

$$\begin{aligned} (4x-1)\log_3 3 &= 3 \\ (4x-1)1 &= 3 \\ 4x &= 4 \\ x &= 1 \end{aligned}$$

1.2.3 Example

$$2^{6x^2+4x} = 7$$

Even though 7 is not a nice power of 2, we still take a logarithm base 2 to simplify the left hand side.

$$\begin{aligned} \log_2 2^{6x^2+4x} &= \log_2 7 \\ (6x^2+4x)\log_2 2 &= \log_2 7 \\ 6x^2+4x &= \log_2 7 \end{aligned}$$

This is looking like a quadratic in x, so we group all the terms on the left hand side and then use the quadratic formula.

$$\begin{aligned} 6x^2+4x-\log_2 7 &= 0 \\ x &= \frac{-4 \pm \sqrt{16+24\log_2 7}}{12} \end{aligned}$$

This is messy, but it is still a reasonable pair of solutions.

1.2.4 Example

$$4^{x^2-2} = 8^{x^2+x-1}$$

We use a logarithm base 2, since 4 and 8 are powers of 2.

$$\begin{aligned}
\log_2 4^{x^2-2} &= \log_2 8^{x^2+x-1} \\
(x^2-2)\log_2 4 &= (x^2+x-1)\log_2 8 \\
2x^2-4 &= 3x^3+3x-3 \\
0 &= x^2+3x+1 \\
x &= \frac{-3 \pm \sqrt{9-4}}{2} = \frac{-3 \pm \sqrt{5}}{2}
\end{aligned}$$

1.2.5 Example

$$\log_4(x^2+5x+4) - \log_4(x+1) = 2$$

We can combine the logarithms to simplify the expression. The difference becomes a quotient.

$$\log_4 \frac{x^2+5x+4}{x+1} = 2$$

We take caution to note that $x \neq -1$, since we would divide by 0. Then we use an exponent base 4 to cancel out the logarithm.

$$\begin{aligned}
4^{\log_4 \frac{x^2+5x+4}{x+1}} &= 4^2 \\
\frac{x^2+5x+4}{x+1} &= 16
\end{aligned}$$

Then we solve this like the rational expression from before.

$$\begin{aligned}
x^2+5x+4 &= 16x+16 \\
x^2-11x-12 &= 0 \\
(x-12)(x+1) &= 0
\end{aligned}$$

We have to reject the $x = -1$ solution as previously noted.

$$x = 12$$

1.2.6 Example

$$\log_3(10-x) - \log_3(x+2) = 1$$

We combine the logarithms into a quotient, and note that $x \neq -2$.

$$\log_3 \frac{10-x}{x+2} = 1$$

We take exponents base 3.

$$\begin{aligned} 3^{\log_3 \frac{10-x}{x+2}} &= 3^1 \\ \frac{10-x}{x+2} &= 3 \\ 10-x &= 3x+6 \\ -4x &= -4 \\ x &= 1 \end{aligned}$$

1.2.7 Example

$$\log_3(2x-1) + \log_3(x+4) = 4$$

We combine the logarithms into a product.

$$\log_3(2x-1)(x+4) = 4$$

We take exponents with base 3.

$$\begin{aligned} (2x-1)(x+4) &= 3^4 = 271 \\ 2x^2+7x-4-271 &= 0 \\ 2x^2+7x+275 &= 0 \\ x &= \frac{-7 \pm \sqrt{49-2\cdot 4\cdot 275}}{4} \end{aligned}$$

There are no solutions, since the square root is negative.

1.2.8 Example

$$\ln(x^2+1) - \ln(x^2-1) = 1$$

We combine the logarithms into a quotient.

$$\ln \frac{x^2+1}{x^2-1} = 1$$

We take exponents with base e, since this is the natural logarithm.

$$\frac{x^2+1}{x^2-1} = e^1 = e$$

To isolate x, we need to clear denominators, then group terms with x^2 and factor out the x^2.

$$\begin{aligned}
x^2 + 1 &= e(x^2 - 1) = ex^2 - e \\
x^2 - ex^2 &= -1 - e \\
x^2(1-e) &= -1 - e \\
x^2 &= \frac{-1-e}{1-e} = \frac{e+1}{e-1} \\
x &= \pm\sqrt{\frac{e+1}{e-1}}
\end{aligned}$$

1.2.9 Example

$$\begin{aligned}
\log_3 x^2 - \log_3(x^2+1) &= 2 \\
\log_3 \frac{x^2}{x^2+1} &= 2 \\
\frac{x^2}{x^2+1} &= 3^2 = 9 \\
x^2 &= 9x^2 + 9 \\
-8x^2 &= 9 \\
x^2 &= \frac{-9}{8}
\end{aligned}$$

There are no solution, since x^2 must be positive.

1.2.10 Example

$$\begin{aligned}
\ln x + \ln x^2 + \ln x^3 &= 7 \\
\ln x \cdot x^2 \cdot x^3 &= 7 \\
\ln x^6 &= 7 \\
x^6 &= e^7 \\
x &= \pm \sqrt[6]{e^7}
\end{aligned}$$

1.3 Isolating variables

As with polynomials and rational function, often we are isolating one variable among many, instead of just solving an equation in one variable. Here are some examples.

1.3.1 Example: solve for a:

$$e^{\frac{a+b}{2}} = 4$$

We take the natural logarithm of both sides.

$$\begin{aligned}
\ln e^{\frac{a+b}{2}} &= \ln 4 \\
\frac{a+b}{2} &= \ln 4 \\
a + b &= 2 \ln 4 \\
a &= 2 \ln 4 - b
\end{aligned}$$

1.3.2 Example: solve for a:

$$\log_2(a^2 - b^2) - \log_2(a + b) = 4$$

Simplify the logarithm into a quotient

$$\log_2 \frac{a^2 - b^2}{a + b} = 4$$

Take exponents with base 2

$$\frac{a^2 - b^2}{a + b} = 2^4 = 16$$

Notice that we can factor the numerator on the left to simplify the expression.

$$\begin{aligned}
\frac{(a-b)(a+b)}{a+b} &= 16 \\
a-b &= 16 \\
a &= 16+b
\end{aligned}$$