Precalculus Review Examples: Algebra

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These examples cover the high-school level technical skills needed to succeed in the study of calculus.

To help emphasize the techniques, we'll be very explicit in these notes with comments on each step in each solution, indicating why we chose to take a certain action.

When working problems yourself, we encourage you to be more explicit and careful in writing up solutions, even to simple problems. Many difficulties are solved just by showing more steps and organizing your work.

Contents

1 Algebra

These examples cover polynomials and rational expressions (one polynomial divided by another). The techniques of understanding polynomals and working with fractions are central.

The main strategies when working with polynomials and rational expressions are:

- Factoring
- Expanding
- Taking sums/differences of fractions to common denominator
- Simplifying nested fractions
- Finding roots of polynomials
- Using the quadratic formula

1.1 Quadratics

We'll start with solving some exaples of quadratic equations. In each of the following, we're solving for the variable x.

1.1.1 Example

$$x^2 - 10x + 25 = 0$$

This is a quadratic. First we try to factor. We want to find a and b with $ab = 25$ and $a + b = -10$. $a = -5$ and $b = -5$ satisfy, so the quadratic factors as:

$$(x - 5)(x - 5) = 0$$

The equation is solved if either factor is 0.

$$\begin{aligned} x - 5 = 0 &\text{ or } x - 5 = 0 \\ x &= 5 \end{aligned}$$

1.1.2 Example

$$3x^2 - 9x - 30 = 0$$

This is also a quadratic. I'll try to factor it, but first we notice that all three coefficients are divisible by 3. I'll factor the 3 out first.

$$3(x^2 - 3x - 10) = 0$$

To simplify, I'll divide both side of the equation by 3.

$$x^2 - 3x - 10 = 0$$

Now we factor. We want a and b with $ab = -10$ and $a + b = -3$. $a = -5$ and $b = 2$ satisfy.

$$(x - 5)(x + 2) = 0$$

The equation holds if either factor is 0.

$$x = 5 \text{ or } x = -2$$

1.1.3 Example

$$x^2 - 25 = 0$$

There are several ways to approach this. First, we add 25 to both sides of the equation.

$$x^2 = 25$$

Then we take square-roots of both sides. We need to remember that the square root can be positive or negative.

$$x = \pm\sqrt{25} = \pm 5$$
$$x = 5 \text{ or } x = -5$$

Alternatively, we could recognize that the original $x^2 - 25$ is a different of squares $x^2 - 5^2$, which is a special form of the quadratic. It factors as:

$$(x - 5)(x + 5) = 0$$

This leads to the same values.

$$x = 5 \text{ or } x = -5$$

1.1.4 Example

$$2x^2 + 5x + 17 = 0$$

We can factor out the 2 to see if the quadratic will factor.

$$2\left(x^2 + \frac{5}{2}x + \frac{17}{2}\right) = 0$$

A factoring would need $ab = \frac{17}{2}$ and $a + b = \frac{5}{2}$. No obvious choices occur, so we use the quadratic formula instead. To work with easier numbers, we use the original coifficients.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ x &= \frac{-5 \pm \sqrt{25 - 8 \cdot 17}}{4} \\ x &= \frac{-5 \pm \sqrt{-111}}{4} \end{aligned}$$

Since there is a negative square root, there are no solutions.

1.1.5 Example

$$4x^2 + 16x + 2 = 0$$

We try to factor our the 4 to see if the quadratic factors nicely.

$$4\left(x^2 + 4 + \frac{1}{2}\right) = 0$$

We need $ab = \frac{1}{2}$ and $a + b = 4$. No immediate solutions occur, so we use the quadratic formula.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ x &= \frac{-16 \pm \sqrt{(16)^2 - 4 \cdot 4 \cdot 2}}{4 \cdot 2} = \frac{-16 \pm \sqrt{256 - 32}}{8} = \frac{-16 \pm \sqrt{224}}{8} \end{aligned}$$

Though this is a final solution, we often want to factor out terms to find a simplier form. We can factor 224 as $16 \cdot 14$, and take the 16 out of the square root as 4 to get:

$$\begin{aligned} x &= \frac{-16 \pm 4\sqrt{14}}{8} = \frac{-4 \pm \sqrt{14}}{2} \\ x &= -2 + \frac{\sqrt{14}}{2} \text{ or } -2 - \frac{\sqrt{14}}{2} \end{aligned}$$

1.2 Higher Degree Polynomials

Quadratics are well understood through factoring and using the quadratic formula. However, we often need to deal with higher order polynomials, where the work can get more difficult.

Looking at cubics, there are three approaches. First, there are two special forms, differences and sums of cubes, which factor easily. Second, cubics always have at least one real root, so we often try to guess one root to reduce to a quadratic. Third, there is a cubic equation, like the quadratic equation. However, it is quite complicated and we rarely rely on it.

Here are some cubic examples. We are asked to solve the following equations for x.

1.2.1 Example

$$x^3 - 27 = 0$$

27 is 3 cubed, so this is a difference of cubes. We use the difference of cubes form from reference materials.

$$(x-3)(x^2+3x+9) = 0$$

The first term gives $x = 3$ as a root, and the second is a quadratic. It doesn't factor nicely, so we use the quadratic formula.

$$x = \frac{-3 \pm \sqrt{3^3 - 4 \cdot 9}}{2} = \frac{-3 \pm \sqrt{-27}}{4}$$

The square root is negative, so there are no roots. Therefore, $x = 3$ is the only soution.

1.2.2 Example

$$x^3 + 64 = 0$$

64 is 4^3, so this is a sum of cubes. We use the sum of cubes form.

$$(x+4)(x^2-4x+16) = 0$$

The first term gives $x = -4$ as a solution. The second term is a quadratic, for which we use the quadratic formula

$$x = \frac{4 \pm \sqrt{16-64}}{2} = \frac{4 \pm \sqrt{-48}}{2}$$

The square root is negative, so there are no roots. Therefore, $x = -4$ is the only solution to the original cubic.

1.2.3 Example

$$x^3 + 2x^2 - 9x + 2 = 0$$

This is not a specific form, but we can try to guess a real root. Trying small integers, we find that $x = 2$ satisfies. That means that $(x - 2)$ must be a factor. Therefore the equation has the form:

$$(x - 2)(ax^2 + bx + c) = 0$$

To find the coefficients a, b and c, we multiply out to recover the original. That means that a must be 1, to get the right cubic term. Then b must be 4 to get the right quadratic term. Finally, c must be -1 to get the right linear term.

$$(x - 2)(x^2 + 4x - 1) = 0$$

We have one root $x = 2$. The remaining piece is a quadratic, and we use the quadratic formula since it doesn't obviously factor.

$$x = \frac{-4 \pm \sqrt{4^2 + 4}}{2} = \frac{-4 \pm \sqrt{12}}{2} = -2 \pm \sqrt{3}$$

Notice in the last step, we factored a factor of 2 out of numerator and denominator. That came out of the square root terms as a factor of 4. This gives us the final two roots, for three total solutions.

$$x = 2 \text{ or } x = -2 + \sqrt{3} \text{ or } x = -2 - \sqrt{3}$$

1.2.4 Example

$$x^3 + 4x^2 - 47x - 210 = 0$$

This is not a special form, so we can try to make some guesses. Guessing small integers is a little more computationally difficult here, but we can find that $x = -5$ is a solution. We factor off $(x = 5)$ to get:

$$(x + 5)(x^2 - x - 42) = 0$$

The $x^2 - x - 42$ term was found in the same way as the previous example, where we find the coefficients of the quadratic piece by multiplying out and matching with the original. This quadratic factors nicely:

$$(x + 5)(x + 6)(x - 7) = 0$$

$$x = -5 \text{ or } x = -6 \text{ or } x = 7$$

1.2.5 Example

$$x^3 + 7x^2 - 3x - 2 = 0$$

This is not a special form. Also, no reasonably small integer guesses help. The only recourse is using the formula. We're not even going to work that out, but a computer approximation gives the following approximate roots, to two decimal places:

$$x = 0.74 \text{ or } x = -0.37 \text{ or } x = -7.37$$

The last example makes the point that already for cubics, exact solutions are difficult to find and approximate solutions are sometime the best we can manage.

The situation just gets more difficult with higher degree polynomials. It becomes more and more difficult to factor and find roots directly. For degree 4, there is a formula to find roots, but it takes several pages to write out. For degree 5 and higher, no formula exists and approximation methods are the only methods remaining.

1.3 Rational Expressions

A rational expression is a fraction where the numerator and denominator are both polynomials. We often want to solve equations with these as well, and we'll give some examples here of the techniques involed.

With rational expressions, we have to be very careful we don't divide by 0. Any solution which leads to 0 in the denominator of the original equation is an invalid solution.

Here are some equations involving rational expressions:

1.3.1 Example

$$\frac{x^2 + 4x - 2}{x^2 - 3x + 7} = 1$$

To get rid of the fraction, we multiply both sides of the equation by $x^2 - 3x + 7$.

$$x^2 + 4x - 2 = 1(x^2 - 3x + 7) = x^2 - 3x + 7$$

Then we want all the terms on one side of the equation. Subtract x^2 from both sides of the equation.

$$4x - 2 = -3x + 7$$

We subtract $-3x + 7$ from both sides.

$$\begin{aligned} 7x - 9 &= 0 \\ x &= \frac{9}{7} \end{aligned}$$

We have a potential solution, but we have to check for division by 0. This x in the original equation gives a denominator of 49 which is not zero, so this is a valid solution.

1.3.2 Example

$$\frac{x^2-4x+3}{x^2+3x-4}=3$$

Instead of worrying about it later, we can check the denominator at the start. Here the denominator factors as $(x-1)(x+4)$. Therefore, $x=1$ and $x=-4$ lead to 0 in the denominator, and are invalid solutions. As long as we avoid these values, we will have valid solutions.

$$x^2-4x+3=3(x^2+3x-4)=3x^2+9x-12$$

We subtract the terms on the right-hand-side from both sides of the equation.

$$-2x^2-13x+15=0$$

This is now a recognizable quadratic, which we solve with the quadratic formula.

$$x=\frac{13\pm\sqrt{13^2-8\cdot 15}}{-4}=\frac{-13\pm\sqrt{49}}{4}=\frac{-13\pm 7}{4}$$

$$x=-5 \text{ or } \frac{-3}{2}$$

Many times we have to manipulate rational expressions that are expressed as several different fractions. The key adjustment here is using common denominator with variable expressions.

1.3.3 Example

$$\frac{1}{x+2}-\frac{1}{2x-2}=4$$

We must find a common denoninator. There are no common factors in $x+1$ and $2x-2$, so the common denominator must be $(x+1)(2x-1)$.

$$\frac{(2x-2)-(x+1)}{(x+1)(2x-2)}=4$$

$$\frac{x-3}{(x+1)(2x-2)}=4$$

$$x-3=4(x+1)(2x-2)=8x^2-8$$

We group all terms on one side of the equation, by subtracting terms as necessary from both sides.

$$-8x^2+x+7=0$$

Now use the quadratic formula.

$$x=\frac{-1\pm\sqrt{197}}{16}$$

$$x=\frac{-1+\sqrt{197}}{16} \text{ or } x=\frac{-1-\sqrt{197}}{16}$$

Neither of these answers are $x=-2$ or $x=1$ which lead to 0 denominators, so these are valid solutions.

1.3.4 Example

$$\frac{x+2}{x-2}+\frac{x-3}{x+1}=2$$

We must use the common denominator $(x+1)(x-2)$.

$$\frac{(x+1)(x+2)+(x-3)(x-2)}{(x-1)(x-2)}=2$$

We simplify the expressions, and multiply by the denominator to clear the fraction.

$$x^2+3x+2+x^2-5x+6=2(x^2-x-2)=2x^2-2x-4$$
$$2x^2-2x+8=2x^2-2x-4$$

We subtract $2x^2-2x$ from both sides of the equation.

$$8=-4$$

We are left with an impossible statement, so there must be no solutions to the original equation.

1.4 Isolating variables

In the following set of examples, instead of solving a single variable equation, we will be isolating one of a number of variables. Often equations are presented to us in this form, and we want to isolate the variable we are most interested in. There are no new methods in this section, just a new setting for the same kind of algebraic manipulations. Be warned, however; these types of questions can be very difficult or impossible, depending on the original situation.

1.4.1 Example: Isolate the variable b:

$$\frac{3a+b}{a^2}+\frac{a}{b+a}=\frac{4}{a}$$

First we note that $a\neq 0$ and $b\neq -a$, so that we don't divide by zero. Then we move to a common denominator

$$\frac{(3a+b)(b+a)+a^3}{a^2b+a^3}=\frac{4}{a}$$

We can multiply both sides of the equation by a and then $ab + a^2$ to cancel off the denominators.

$$\begin{aligned}\frac{3a^2 + 4ab + b^2 + a^3}{ab + a^2} &= 4 \\ a^3 + 3a^2 + 4ab + b^2 &= 4(ab + a^2) = 4ab + 4a^2\end{aligned}$$

We can subtraction $4ab$ from both sides of the equation.

$$a^3 + 3a^2 + b^2 = 4a^2$$

Next we group like variables on either side of the equation, isolating b^2 since we want to isolate b.

$$\begin{aligned}b^2 &= a^2 - a^3 \\ b &= \pm\sqrt{a^2 - a^3}\end{aligned}$$

1.4.2 Example: Isolate the variable a:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{ab} = 1$$

We go to the common denominator of ab.

$$\frac{b + a + 1}{ab} = 1$$

We multiply both sides by ab to clear the denominator.

$$a + b + 1 = ab$$

We want to isolate a, so we group all the terms with a on the left hand side by adding and subtracting terms from both sides.

$$\begin{aligned}a - ab &= -b - 1 \\ a(1 - b) &= -b - 1 \\ a &= \frac{-b - 1}{1 - b} = \frac{b + 1}{b - 1}\end{aligned}$$

1.4.3 Example: Isolate the variable Q:

(This one is very tricky. It is included to show how to think about applying the same techniques as before in more and more complicated situations.)

$$\left(P + \frac{Q}{S}\right)\left(P^2 - S^2\right) = \frac{1}{Q}$$

Multiply both sides by Q.

$$Q\left(P+\frac{Q}{S}\right)\left(P^2-S^2\right)=1$$

Expand the multiplication on the left hand side.

$$\left(QP+\frac{Q^2}{S}\right)\left(P^2-S^2\right)=1$$

$$QP^3+\frac{Q^2P^2}{S}-QPS^2-Q^2S=1$$

To simplify a little, lets clear the denominators by multpliying everything on both sides by S.

$$QSP^3+Q^2P^2-QPS^3-Q^2S^2=S$$

We group terms with respect to degree of Q, since that's the variable we wish to isolate. We also bring the S over to the left hand side.

$$(P^2-S^2)Q^2+(P^3S-PS^3)Q-S=0$$

Though strange to look at, this is a quadratic in Q with $a=P^2-S^2$, $b=P^3S-PS^3$ and $c=-1$. We can use the quadratic formula to get the following complicated solution.

$$Q=\frac{PS^3-P^3S\pm\sqrt{(P^3S-PS^3)^2-4(P^2-S^2)S}}{4(P^2-S^2)}$$

We could try to simplify further, but this is a valid solution. We don't know whether this quadratic has real roots without giving values for P and S.