Problem Set 29: Curve Sketching

Key skills: Curve Sketching

Practice Problems

Sketch the following functions, using all the function properties discussed in class.

a) $f(x) = \dfrac{x^2}{1+x^2}$

b) $f(x) = \ln(x^2+4)$

c) $f(x) = e^{-x}\sin(x+\pi)$

d) $f(x) = x^2 + \dfrac{1}{x^3}$

Answers

a) The domain is $\mathbb{R}$. The range is $[0, 1)$. The function has even symmetry. There is an x and y-intercept at $(0, 0)$ and no other intercepts. The function is always positive. The limit as $x \to \pm\infty$ is 1, so $y = 1$ is a horizontal asymptote in both the positive and negative directions.

The first derivative is

$$f'(x) = \frac{2(1 + x^2) - x^2(2x)}{(1 + x^2)^2} = \frac{2x}{(1 + x^2)^2)}$$

This has a root at $x = 0$. The function is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$. $x = 0$ is a local minimum.

The second derivative is

$$f''(x) = \frac{2(1 + x^2)^2 - 2x(2x)(2)(1 + x^2)}{(1 + x^2)^4} = \frac{2(1 + x^2) - 8x^2}{(1 + x^2)^3} = \frac{2 - 6x^2}{(1 + x^2)^3}$$

This has roots at $x = \pm\sqrt{\frac{1}{3}}$. The function is concave down on $\left(-\infty, -\sqrt{\frac{1}{3}}\right)$, concave up on $\left(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right)$ and concave down on $\left(\sqrt{\frac{1}{3}}, \infty\right)$. Both roots are inflection points.

b) The domain is $\mathbb{R}$. The range is $[\ln 4, \infty)$. The function has even symmetry. There is a y-intercept at $(0, \ln 4)$. There are no x intercepts. The function is always positive. The limit as $x \to \pm\infty$ is ∞, so there are no asymptotes.

The first derivative is

$$f' = \frac{2x}{x^2 + 4}$$

This has a root at $x = 0$. The function is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$. $x = 0$ is a local minimum.

The second derivative is

$$f''(x) = \frac{2(x^2 + 4) - 2x(2x)}{(x^2 + 4)^2} = \frac{8 - 2x^2}{(x^2 + 4)^2}$$

This has a root at $x = \pm 2$. The function is concave down on $(-\infty, -2)$, concave up on $(-2, 2)$ and concave down on $(2, \infty)$. Both roots are inflection points.

c) The domain is $\mathbb{R}$ and the range is $\mathbb{R}$. There is no symmetry. There is a y-intercept at $(0, 0)$ and infinitely many x-intercepts at all multiples of π. The limit as $x \to \infty$ is 0, so $y = 0$ is a horizontal asymptote in the positive direction. The limit as $x \to -\infty$ does not exist.

The first derivative is

$$f'(x) = -e^{-x}\sin(x+\pi) + e^{-x}\cos(x+\pi) = e^{-x}(\cos(x+\pi) - \sin(x+\pi)$$

This has infinitely many roots, whenever sin and cos are equal. (Those values are slightly tricky to calculate). All of these critical points are maxima or minima, as the sinusoidal part of the functions oscillates up and down.

The second derivative is

$$f''(x) = e^{-x}\sin(x+\pi) - 2e^{-x}\cos(x+\pi) - e^{-x}\sin(x+\pi) = -2e^{-x}\cos(x+\pi)$$

There are infinitely many roots, at all odd multiples of $\pi/2$. All these roots are inflection points, as the sinusoidal part of the function switches between concave up to concave down.

d) The domain is all $x \neq 0$. The range is $\mathbb{R}$. There are no y-intercepts. There are no x-intercepts. There is no symmetry. The limits as $x \to 0$ are $\pm\infty$, so there is a vertical asymptote at $x = 0$. The limits as $x \to \pm\infty$ are ∞ so there are no horizontal asymptotes.

The first derivative is

$$f'(x) = 2x - \frac{1}{3x^4}$$

It has a root at $x = \sqrt[5]{\frac{1}{6}}$. The function is decreasing on $(-\infty, 0)$, decreasing on $\left(0, \sqrt[5]{\frac{1}{6}}\right)$ and increasing on $\left(\sqrt[5]{\frac{1}{6}}, \infty\right)$. $x = \sqrt[5]{\frac{1}{6}}$ is a local minimum.

The second derivative is

$$f'' = 2 + \frac{1}{12x^5}$$

It has a root at $x = \sqrt[5]{\frac{-1}{24}}$. The function is concave up on $\left(-\infty, \sqrt[5]{\frac{-1}{24}}\right)$. The function is concave down on $\left(\sqrt[5]{\frac{-1}{24}}, 0\right)$. The function is concave up on $(0, \infty)$. $x = \sqrt[5]{\frac{-1}{24}}$ is an inflection point.