Problem Set 28: Marginal Analysis

Part of the larger field of "Cost-Benefit Analysis".

Key skills: Optimization, Intersections

Practice Problems

For each of the following pairs of cost and benefit functions, find the point of maximum net benefit, and the range(s) of production rates with a positive net benefit. Determine whether production should be increased, decreased, or neither at production levels $x = 1, 2, 3$. (Recall that x is the production rate, $C(x)$ is the cost of producing at rate x, and $B(x)$ is the benefit (income from sale) obtained from production at rate x.)

In most of these cases you'll need to have a computer calculate the final solutions. Your main job is to come up with the equations for the computer to solve. One way is to ask Wolfram Alpha to "solve" the function (for example: `solve 0=x^3-8x^2+4x-25`). You may need to click "Approximate forms" to get a numerical answer.

a) $C(x) = x^2 + 6$
$B(x) = -x^2 + 8x$

b) $C(x) = \dfrac{x^2}{8}$
$B(x) = \sqrt{x}$

c) $C(x) = \dfrac{x^2 + 1}{4}$
$B(x) = \begin{cases} \frac{x^2}{2} & x \leq 2 \\ 2x - 2 & x > 2 \end{cases}$

d) $C(x) = e^{x/5}$
$B(x) = 2\ln(x + 1)$

e) $C(x) = x + 4$
$B(x) = \dfrac{10x}{x + 1}$

Answers

Define $N(x) = B(x) - C(x)$ *to be the net benefit.*

a) $N(x) = -2x^2 + 8x - 6 = -2(x-1)(x-3)$

Net benefit is positive for $1 < x < 3$.

$N'(x) = -4x + 8$

The derivative is 0 at $x = 2$, so since $N(x)$ is concave-down (negative x^2 term) this is the point of maximum net benefit. $N'(x)$ is positive at $x = 1$, zero at $x = 2$, and negative at $x = 3$, so production should increase, stay the same, and decrease at these levels, respectively.

b) $N(x) = \sqrt{x} - \dfrac{x^2}{8}$

Net benefit is positive for $0 < x < 4$.

$$N'(x) = \frac{1}{2\sqrt{x}} - \frac{x}{4}$$

The derivative is 0 at $x = 2^{2/3} \approx 1.587$, so since $N(x)$ is concave-down this is the point of maximum net benefit. $N'(x)$ is positive at $x = 1$, so production should increase from there; it is negative at $x = 2$ and 3, so production should be decreased from those values.

c) $N(x) = \begin{cases} \frac{x^2}{2} - \frac{x^2+1}{4} & x \le 2 \\ 2x - 2 - \frac{x^2+1}{4} & x > 2 \end{cases}$

or

$$N(x) = \begin{cases} \frac{3x^2-1}{4} & x \le 2 \\ \frac{-x^2+8x-9}{4} & x > 2 \end{cases}$$

Net benefit is positive for $\frac{1}{\sqrt{3}} < x < 4 + \sqrt{7}$ (approximately $0.577 < x < 6.646$).

$$N'(x) = \begin{cases} \frac{3x}{2} & x \le 2 \\ -\frac{x}{2} + 2 & x > 2 \end{cases}$$

This derivative is 0 at $x = 0$ and $x = 4$; only $x = 4$ has positive net benefit, so this must be the maximum. $N'(x)$ is positive at $x = 1, 2, 3$ so production should be increased in each of those cases.

d) $N(x) = 2\ln(x+1) - e^{x/5}$

Net benefit is positive for approximately $(0.798 < x < 7.179)$.

$$N'(x) = \frac{2}{x+1} - \frac{1}{5}e^{x/5}$$

This derivative is 0 at $x \approx 3.736$, so that production gives maximum net benefit. $N'(x)$ is positive at $x = 1, 2, 3$ so production should be increased in each of those cases.

e) $N(x) = \dfrac{10x}{x+1} - x - 4 = \dfrac{10x - (x^2 + 5x + 4)}{x+1} = \dfrac{-x^2 + 5x - 4}{x+1} = \dfrac{-(x-1)(x-4)}{x+1}$

Net benefit is positive for $1 < x < 4$.

$$N'(x) = \frac{(-2x+5)(x+1) - (-x^2 + 5x - 4)(1)}{(x+1)^2} = \frac{-x^2 - 2x + 9}{(x+1)^2}$$

This derivative is 0 at $x = \sqrt{10} - 1 \approx 2.162$, so since $N(x)$ is concave-down this is the point of maximum net benefit. $N'(x)$ is positive at $x = 1$ and 2, so production should increase from there. $N'(x)$ is negative at $x = 3$ so production from there should be decreased.