Problem Set 27: Optimization

Key skills: Optimization

Practice Problems

1. If there is a square prism with height, width h and length 1 m, determine the maximum volume of the prism if the surface area is 1 m^2.

2. The area of an ellipse with semi-axes a and b is πab. Find the maximum area under the constraint $a + b = 1$. Then find the maximum area under the constraint $a^2 + b^2 = 1$.

3. Two circles have centres which are exactly 1 m apart. The circles touch each other but do not overlap. What is the minimum of the sum of the areas of the two circles in this situation? *(Hint: The radii do not have to be the same!)*

Answers

1. Surface area is $2h^2 + 4hl = 1$. We solve for l and subsitute into the volume formula $V = h^2 l$. This gives

$$V = \frac{h - 2h^3}{4} \qquad V'(h) = \frac{1}{4} - \frac{6h^2}{4}$$

Solving for the critical points gives $h = \frac{1}{\sqrt{6}}$. We check that this is a maximum by seeing that the sign of the derivative changes from positive to negative. Then $l = \frac{1}{\sqrt{6}}$ as well, and we conclude that the cube has the maximum volume.

2. Process in both parts is the same as the previous question, using the constraint to remove a variable, and using the first derivative and critical points to find the maximum. The maximum area for the first part is $\pi/4$, and for the second part is $\pi/2$.

3. If r_1 and r_2 are the two radii, then the areas are πr_1^2 and πr_2^2 with the restriction that $r_1 + r_2 = 1$. Therefore $r_2 = 1 - r_1$. The total area is

$$A = \pi r_1^2 + \pi(1 - r_1)^2 = \pi(1 - 2r_1 + 2r_1^2)$$

If we take the derivative and optimize, we find a local minimum at $r_1 = \frac{1}{2}$. Therefore, the total area is minimized when both radii are $1/2$ and the minimum sum of the areas of both circles is $\pi/2$ units squared.