Problem Set 26: Extrema

Key skills: Derivatives, Critical points, Slopes

Practice Problems

Find the x values of the critical points of each of the following functions, and classify each as a local minimum, local maximum, or neither. As a bonus, determine if any of the critical points are *global* minima or maxima.

Check your answers by plotting the functions with Desmos (https://www.desmos.com) or other graphing software.

a) $f(x) = (x-2)^3$ *b*) $f(x) = xe^{-x}$ *c*) $f(x) = x^2e^{-x}$ *d*) $f(x) = xe^{-x^2}$

e) $f(x) = \sqrt{x^2+4}$ *f*) $f(x) = x\ln x$ *g*) $f(x) = \dfrac{x^3}{3} - 4x + 4$

h) $f(x) = \sin(x^2)$ *i*) $f(x) = \sin^2 x$ *j*) $f(x) = \dfrac{x^2+6x-1}{x^2+2}$

Answers

Remember that it's very useful to simplify and factor the derivative wherever possible (or at least convenient), since that makes the zeroes much easier to find. To determine whether an extremum is global or just local, remember to consider the limits as x goes to $\pm\infty$.

(a) $f'(x) = 3(x-2)^2 = 0 \implies x = 2$. $f'(x)$ is positive on both sides of $x = 2$ the function is increasing everywhere, so $x = 2$ is neither a minimum nor a maximum.

(b) $f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x} = 0 \implies x = 1$. $f'(x)$ is positive for $x < 1$, and negative for $x > 1$, so this is a (global) maximum.

(c) $f'(x) = 2xe^{-x} - x^2e^{-x} = (2x - x^2)e^{-x} = x(2-x)e^{-x} = 0 \implies x = 0$ or $x = 2$. $f'(x)$ is negative for $x < 0$, positive for $0 < x < 2$, and negative for $x > 2$, so $x = 0$ is a (global) minimum and $x = 2$ is a (local) maximum.

(d) $f'(x) = e^{-x^2} - 2x^2e^{-x^2} = (1-2x^2)e^{-x^2} = 0 \implies x = \pm\frac{1}{\sqrt{2}}$ (roughly ± 0.707). $f'(x)$ is negative for $x < -\frac{1}{\sqrt{2}}$, positive for $\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$, and negative for $x > \frac{1}{\sqrt{2}}$, so $x = -\frac{1}{\sqrt{2}}$ is a (global) minimum and $x = \frac{1}{\sqrt{2}}$ is a (global) maximum.

(e) $f'(x) = \frac{x}{\sqrt{x^2+4}} = 0 \implies x = 0$. $f'(x)$ is negative for $x < 0$ and positive for $x > 0$, so $x = 0$ is a (global) minimum.

(f) $f'(x) = \ln x + 1 = 0 \implies x = e^{-1} = \frac{1}{e}$ (roughly 0.367). $f'(x)$ is negative for $x < \frac{1}{e}$ and positive for $x > \frac{1}{e}$, so $x = \frac{1}{e}$ is a (global) minimum.

(g) $f'(x) = x^2 - 4 = (x-2)(x+2) = 0 \implies x = \pm 2$. $f'(x)$ is positive for $x < -2$, negative for $-2 < x < 2$, and positive for $x > 2$, so $x = -2$ is a (local) maximum and $x = 2$ is a (local) minimum.

(h) $f'(x) = 2x\cos(x^2) = 0 \implies x = 0$ or $x = \pm\sqrt{(n+\frac{1}{2})\pi}$, $n = 0, 1, 2, \ldots$. $f'(x)$ is negative just to the left of $x = 0$ and positive just to the right, so $x = 0$ is a (local) minimum. Moving out from here, $f'(x)$ flips sign after each critical point, so the critical points alternate between maxima and minima. All extrema are local.

(i) $f'(x) = 2\sin x \cos x = 0 \implies x = n\left(\frac{\pi}{2}\right)$, $n = 0, \pm 1, \pm 2, \ldots$. $f'(x)$ is negative just to the left of $x = 0$ and positive just to the right, so $x = 0$ is a minimum. Moving out from here, $f'(x)$ flips sign after each critical point, so the critical points alternate between maxima and minima. All extrema are local.

(j) $f'(x) = \frac{(2x+6)(x^2+2)-(x^2+6x-1)(2x)}{(x^2+2)^2} = \frac{-6x^2+6x+12}{(x^2+2)^2} = \frac{-6(x+1)(x-2)}{(x^2+2)^2} = 0 \implies x = -1$ or $x = 2$. $f'(x)$ is negative for $x < -1$, positive for $-1 < x < 2$, and negative for $x > 2$, so $x = -1$ is a (global) minimum, and $x = 2$ is a (global) maximum.