Problem Set 20: Sigma Notation

Key skills: Sigma notation for sums, manipulating sums, combining sums, manipulating indices

Practice Problems

1) Write out each of the following sums in full. (Questions a, b, c, h, and i demonstrate how shifting indices works. Question f shows why you can factor constants out of a sum.)

$a)\ \sum_{k=1}^{4} k \qquad b)\ \sum_{k=2}^{5}(k-1) \qquad c)\ \sum_{k=3}^{6}(k-2) \qquad d)\ \sum_{k=2}^{5} k \qquad e)\ \sum_{j=1}^{4} j \qquad f)\ \sum_{k=1}^{4} 2k$

$g)\ \sum_{k=1}^{4} 1 \qquad h)\ \sum_{k=0}^{2} k^2 \qquad i)\ \sum_{k=1}^{3}(k-1)^2 \qquad j)\ \sum_{k=1}^{3}(k^2-1) \qquad k)\ \sum_{k=1}^{3}(k^3-4k)$

2) For each of these sums, take out the first three terms and use sigma notation for the rest.

$a)\ \sum_{k=1}^{10}\frac{k^2}{2} \qquad b)\ \sum_{k=1}^{5}(2k-1) \qquad c)\ \sum_{k=1}^{8}(2k^2+k) \qquad d)\ \sum_{k=3}^{9} k$

3) Combine each pair of sums into a single sigma-notation expression, shifting indices where necessary. *(Tip: When shifting indices of a sum, check it by writing out the first two or three terms; the original and "shifted" versions should work out the same.)*

$a)\ \sum_{k=1}^{12}3k+\sum_{k=1}^{12}5k^2 \qquad b)\ \sum_{k=1}^{7}2k+\sum_{k=3}^{9}3k \qquad c)\ \sum_{k=2}^{4}4k^2+\sum_{k=2}^{4}2k^2 \qquad d)\ \sum_{k=3}^{25}(k^2+2k)+\sum_{k=1}^{23}k+2$

4) Demonstrate the following formulas for the given values of n.

$a)\ \sum_{k=1}^{n} 1 = n \quad \text{for } n=5$

$b)\ \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \text{for } n=6 \text{ and } n=7$

$c)\ \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{for } n=4$

$d)\ \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2 \quad \text{for } n=3$

Answers

Part 1:

a) $\displaystyle\sum_{k=1}^{4} k = 1 + 2 + 3 + 4$

b) $\displaystyle\sum_{k=2}^{5} (k-1) = (2-1) + (3-1) + (4-1) + (5-1) = 1 + 2 + 3 + 4$

c) $\displaystyle\sum_{k=3}^{6} (k-2) = (3-2) + (4-2) + (5-2) + (6-2) = 1 + 2 + 3 + 4$

d) $\displaystyle\sum_{k=2}^{5} k = 2 + 3 + 4 + 5$

e) $\displaystyle\sum_{j=1}^{4} j = 1 + 2 + 3 + 4$

f) $\displaystyle\sum_{k=1}^{4} 2k = 2(1) + 2(2) + 2(3) + 2(4) = 2 + 4 + 6 + 8$

g) $\displaystyle\sum_{k=1}^{4} 1 = 1 + 1 + 1 + 1$

h) $\displaystyle\sum_{k=0}^{2} k^2 = 0^2 + 1^2 + 2^2 = 0 + 1 + 4$

i) $\displaystyle\sum_{k=1}^{3} (k-1)^2 = (1-1)^2 + (2-1)^2 + (3-1)^2 = 0^2 + 1^2 + 2^2 = 0 + 1 + 4$

j) $\displaystyle\sum_{k=1}^{3} (k^2-1) = ((1)^2-1) + ((2)^2-1) + ((3)^2-1) = (1-1) + (4-1) + (9-1) = 0 + 3 + 8$

k) $\displaystyle\sum_{k=1}^{3} (k^3-4k) = (1^3-4(1)) + (2^3-4(2)) + (3^3-4(3)) = (1-4) + (8-8) + (27-12) = 3 + 0 + 15$

Part 2:

a) $\displaystyle\sum_{k=1}^{10} \frac{k^2}{2} = \frac{1}{2} + \frac{4}{2} + \frac{9}{2} + \sum_{k=4}^{10} \frac{k^2}{3}$

b) $\displaystyle\sum_{k=1}^{5} (2k-1) = 1 + 3 + 5 + \sum_{k=4}^{5} (2k-1)$

c) $\displaystyle\sum_{k=1}^{8} (2k^2+k) = 3 + 10 + 21 + \sum_{k=4}^{8} (2k^2+k)$

d) $\displaystyle\sum_{k=3}^{9} k = 3 + 4 + 5 + \sum_{k=6}^{9} k$

Part 3:

a) $\displaystyle\sum_{k=1}^{12} 3k \,+\, \sum_{k=1}^{12} 5k^2 = \sum_{k=1}^{12} (3k+5k^2)$

b) $\displaystyle\sum_{k=1}^{7} 2k \,+\, \sum_{k=3}^{9} 3k = \sum_{k=3}^{9} 2(k-1) \,+\, \sum_{k=3}^{9} 3k = \sum_{k=3}^{9} 5k-2$

or: $\displaystyle\sum_{k=1}^{7} 2k \,+\, \sum_{k=3}^{9} 3k = \sum_{k=1}^{7} 2k \,+\, \sum_{k=1}^{7} 3(k+2) = \sum_{k=1}^{7} 5k+2$

(Convince yourself these are the same result by writing out the first few terms!)

c) $\displaystyle\sum_{k=2}^{4} 4k^2 \,+\, \sum_{k=2}^{4} 2k^2 = \sum_{k=2}^{4} 6k^2$

d) $\displaystyle\sum_{k=3}^{25} (k^2+2k) \,+\, \sum_{k=1}^{23} k+2 = \sum_{k=3}^{25} (k^2+2k) \,+\, \sum_{k=3}^{25} k = \sum_{k=3}^{25} (k^2+3k)$

(You could also shift the first sum instead, but this is simpler.)

Part 4:

a) LHS: $\sum_{k=1}^{5} 1 = 1 + 1 + 1 + 1 + 1 = 5$

RHS: $n = 5 \quad \checkmark$

b) LHS: $\sum_{k=1}^{6} k = 1 + 2 + 3 + 4 + 5 + 6 = 21$

RHS: $\frac{6(6+1)}{2} = 21 \quad \checkmark$

LHS: $\sum_{k=1}^{7} k = 1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$

RHS: $\frac{7(7+1)}{2} = 28 \quad \checkmark$

c) LHS: $\sum_{k=1}^{4} k^2 = 1 + 4 + 9 + 16 = 30$

RHS: $\frac{4(4+1)(2(4)+1)}{6} = 30 \quad \checkmark$

d) LHS: $\sum_{k=1}^{3} k^3 = 1 + 8 + 27 = 36$

RHS: $\left(\frac{3(3+1)}{2}\right)^2 = 36 \quad \checkmark$