Problem Set 4: Operations on Functions

Key Skills: Properties of Functions, Inversion, Composition

Practice Problems

Invert the following functions. To do so, give a reason for the invertibility of the function, restricting the domain if necessary. State the domain and range of the function, as well as the domain of the inverse. (You don't have to calculate the range of c), which is $(0,1)$. You may assume that d), e) and f) are monotonic, since monotonicity is difficult to establish with our current tools.)

Demonstrate that you have the correct inverse by finding $f \circ f^{-1}(x)$ and $f^{-1} \circ f(x)$ on the appropriate domains.

$a)$ $f(x) = \dfrac{1}{e^x}$ $b)$ $f(x) = \sqrt{x^2+1}$ $c)$ $f(x) = \dfrac{1}{1+\frac{1}{(e^x)^2}}$

$d)$ $f(x) = \dfrac{4x-1}{2x+3}$ $e)$ $f(x) = \dfrac{ax+b}{cx+d}$ $f)$ $f(x) = \dfrac{e^x}{1+2e^x}$

$g)$ $f(x) = \ln\left(\dfrac{x}{x+1}\right)$ $h)$ $f(x) = \dfrac{3e^x}{1-2e^x}$

Answers

(a) This function is defined on all of $\mathbb{R}$, and its range is $(0, \infty)$. The range is calculated by inspection: the exponential e^x is always positive, so its reciprocal is as well. e^x can be arbitrarily large or arbitrarily close to zero, so its reciprocal can realize any positive real number. f is a decreasing function, so it is invertible on its domain. Its inverse is defined on $(0, \infty)$ and has the form

$$f^{-1}(x) = -\ln(x)$$

(b) This function is defined on all of $\mathbb{R}$ and its range is $[1, \infty)$. The range is calculated by observing that $x^2 \geq 0$, so the term in the square root is ≥ 1. f a decreasing function on $(-\infty, 0)$ and an increasing functionon $(0, \infty)$, so we restrict its domain to $[0, \infty)$ to invert it. The range remains the same. Its inverse is defined on $[1, \infty)$ and has the form

$$f^{-1}(x) = \sqrt{x^2 - 1}$$

Note that by rearranging $y = \sqrt{x^2 + 1}$ into standard conic form, you can show that $f(x)$ is the top half of a hyperbola.

(c) This function is defined on all of $\mathbb{R}$ and its range is $(0, 1)$ as given. The function is increasing, so it is invertible. Its inverse is defined on $(0, 1)$ and has the form

$$f^{-1}(x) \ \ln \sqrt{\frac{1}{\frac{1}{x} - 1}}$$

(d) This function is defined on all $\mathbb{R}$ except $\frac{-3}{2}$. Its range is all $\mathbb{R}$ except 2. The range is difficult to calculate directly, but can be infered by looking at extreme values. Near the undefined value, the function gets arbitrarily large. For large x, the function gets arbitrarily close to 2. f is an increasing function that doesn't have conflicting values on either side of its asymptote, so it is invertible. The inverse is defined on all $\mathbb{R}$ except 2 and has the form

$$f^{-1}(x) = \frac{3x + 1}{4 - 2x}$$

(e) This function is the generalization of the previous function, with similar properties. It is monotonic, hence invertible, if $ad - bc \neq 0$. In that case, the domain is all $\mathbb{R}$ except $\frac{-d}{c}$ and the range is all $\mathbb{R}$ except $\frac{a}{c}$. The inverse is defined on all $\mathbb{R}$ except a/c and has the form

$$f^{-1}(x) = \frac{-dx + b}{cx - a}$$

Note this is an interesting case since all function of this form (with the condition $ad - bc \neq 0$) have inverses of essentially the same form. These function are called Möbius transformation; they are important in several areas of mathematics, particularly in complex analysis.

(f) The domain is all $\mathbb{R}$. The range is $\left(0, \frac{1}{2}\right)$. The range is difficult to calculate, but can be seen by looking at extreme values. For large negative x, the function is arbitrarily close to 0. For large x the function is arbitrarily close to $\frac{1}{2}$. For the rest of the function, it remains between those two values. The function is monotonic, hence invertible. The inverse is defined on $\left(0, \frac{1}{2}\right)$ and has the form

$$f^{-1}(x) = \ln\left(\frac{x}{1 - 2x}\right)$$

(g) When calculating the domain, remember that a fraction is positive when the numerator and denominator have the same sign, i.e. both positive or both negative; check both cases separately. The domain is all $\mathbb{R}$ except $[-1, 0]$. The range can again be seen by looking at extreme values, and at the behaviour near the boundaries. The function is monotonic, hence invertible. The inverse is defined for all $\mathbb{R}$ except 0, and has the form

$$f^{-1}(x) = \frac{e^x}{1 - e^x}$$

(h) The domain of f is all real except $\ln \frac{1}{2}$. The domain of f^{-1} is the union of the invervals $(-\infty, 0)$ and $\left(\frac{3}{2}, \infty\right)$. The inverse has the form

$$f^{-1}(x) = \ln\left(\frac{x}{2x+3}\right)$$