Problem Sets for Math 200

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Problem Set 1: Background Review

Key Skills: Algebra, Trigonometry, Fractions, Exponents, Logarithms.

Let x, y, a, b be real numbers, positive and non-zero when necessary. Determine whether the following statements are true or false.

$\frac{a}{b} + \frac{a}{c}$	$=$	$\frac{a}{b+c}$	T	F
$(x-y)^2$	$=$	$x^2 - y^2$	T	F
$(2^a)^b$	$=$	2^{a+b}	T	F
$\frac{1}{\frac{1}{x}}$	$=$	$\frac{1}{x}$	T	F
x^{-2}	$=$	$\frac{x}{2}$	T	F
2^{3x}	$=$	6^x	T	F
$2^x 2^y$	$=$	2^{xy}	T	F
$\frac{1}{\sin x}$	$=$	$\sec x$	T	F
$\frac{x+1}{x^2-1}$	$=$	$\frac{1}{x} - 1$	T	F
$\frac{1}{x} + 1$	$=$	$\frac{x}{x+1}$	T	F
$\sqrt{x^2 - x^4}$	$=$	$x^2\sqrt{1-x^2}$	T	F

$$\frac{a}{c} - \frac{b}{d} = \frac{a-b}{c-d} \qquad T \qquad F$$

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ac}{bd} \qquad T \qquad F$$

$$x^3 - y^3 = (x-y)^3 \qquad T \qquad F$$

$$b\log_2 a = (\log_2 a)^b \qquad T \qquad F$$

$$\log_2 \frac{a}{b} = \frac{\log_2 a}{\log_2 b} \qquad T \qquad F$$

$$(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y}) = x + y \qquad T \qquad F$$

$$\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{y} \qquad T \qquad F$$

$$\sin 2x = 2\sin x \qquad T \qquad F$$

$$\sin\left(x + \frac{\pi}{2}\right) = \sin x + \sin\frac{\pi}{2} \qquad T \qquad F$$

$$\sin^2 x = 1 + \cos^2 x \qquad T \qquad F$$

$$\tan^2 x \cot^2 x = \sin^2 x \qquad T \qquad F$$

$$\frac{1}{1+\frac{1}{x}} = \frac{x+1}{x} \qquad T \qquad F$$

$$\frac{1}{x^{-\frac{1}{2}}} = \frac{x}{2} \qquad T \qquad F$$

Answers

Let x, y, a, b be real numbers, positive and non-zero when necessary. Review the solutions to this true-false quiz. For each entry, construct a different right-side to the equation to fix the error.

$\frac{a}{b}+\frac{a}{c}$	$=$	$\frac{a}{b+c}$	T	F	$\frac{ac+ab}{bc}$
$(x-y)^2$	$=$	x^2-y^2	T	F	$x^2-2xy+y^2$
$(2^a)^b$	$=$	2^{a+b}	T	F	2^{ab}
$\frac{1}{\frac{1}{x}}$	$=$	$\frac{1}{x}$	T	F	x
x^{-2}	$=$	$\frac{x}{2}$	T	F	$\frac{1}{x^2}$
2^{3x}	$=$	6^x	T	F	8^x
$2^x 2^y$	$=$	2^{xy}	T	F	2^{x+y}
$\frac{1}{\sin x}$	$=$	$\sec x$	T	F	$\csc x$
$\frac{x+1}{x^2-1}$	$=$	$\frac{1}{x}-1$	T	F	$\frac{1}{x-1}$
$\frac{1}{x}+1$	$=$	$\frac{x}{x+1}$	T	F	$\frac{1+x}{x}$
$\sqrt{x^2-x^4}$	$=$	$x^2\sqrt{1-x^2}$	T	F	$x\sqrt{1-x^2}$

$\frac{a}{c} - \frac{b}{d}$	$=$	$\frac{a-b}{c-d}$	T	F	$\frac{ad-bc}{cd}$
$\frac{\frac{a}{b}}{\frac{c}{d}}$	$=$	$\frac{ac}{bd}$	T	F	$\frac{ad}{bc}$
$x^3 - y^3$	$=$	$(x-y)^3$	T	F	$x^3 - 3x^2y + 3xy^2 - y^3$
$b\log_2 a$	$=$	$(\log_2 a)^b$	T	F	$\log_2(a^b)$
$\log_2 \frac{a}{b}$	$=$	$\frac{\log_2 a}{\log_2 b}$	T	F	$\log_2 a - \log_2 b$
$(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})$	$=$	$x+y$	T	F	$x-y$
$\sqrt{\frac{x}{y}}$	$=$	$\frac{\sqrt{x}}{y}$	T	F	$\frac{\sqrt{x}}{\sqrt{y}}$
$\sin 2x$	$=$	$2\sin x$	T	F	$2\sin x\cos x$
$\sin\left(x+\frac{\pi}{2}\right)$	$=$	$\sin x + \sin\frac{\pi}{2}$	T	F	$\cos x$
$\sin^2 x$	$=$	$1+\cos^2 x$	T	F	$1-\cos^2 x$
$\tan^2 x\cot^2 x$	$=$	$\sin^2 x$	T	F	1
$\frac{1}{1+\frac{1}{x}}$	$=$	$\frac{x+1}{x}$	T	F	$\frac{x}{x+1}$
$\frac{1}{x^{-\frac{1}{2}}}$	$=$	$\frac{x}{2}$	T	F	$\sqrt{x}$

Problem Set 2: Conics and Loci

Key skills: Analytic Geometry, Intersection

Practice Problems

Write the equations of the following lines and conics. Assume all conics (except the line) are centered at the origin, i.e. have equations in standard form. Check your answers by confirming that the given points solve the equations you find.

(a) The line through $(0, 1)$ with slope 3.
(b) The parabola through $(1, 7)$.
(c) The circle through $(2, -4)$.
(d) The ellipse through $(1, 1)$ and $(0, 6)$.

Find the intersection point(s) of the following pairs of lines and/or conics. Check your answers by confirming that the intersection point(s) satisfy the equations of both shapes.

(a) The line through $(-1, -2)$ and $(0, 0)$ and the parabola through $(-1, 4)$.
(b) The parabola through $(3, 3)$ and the hyperbola through $(2, 0)$ and $(4, 4)$.
(c) The line through $(-1, -1)$ with slope 2 and the hyperbola through $(2, 0)$ and $(-3, -1)$.
(d) The line through $(3, 1)$ with slope $-1/2$ and the ellipse centered and the origin which passes through $(-3, 0)$ and $(-2, 2)$.

Answers

Equations:

(a) $y = 3x + 1$
(b) $y = 7x^2$
(c) $x^2 + y^2 = 20$
(d) $\frac{35x^2}{36} + \frac{y^2}{36} = 1$

Intersections:

(a) The line has equation $y = 2x$ and the parabola has equation $y = 4x^2$. The intersection points are $(0, 0)$ and $\left(\frac{1}{2}, 1\right)$.
(b) The parabola has equation $y = \frac{x^2}{3}$ and the hyperbola has equation $\frac{x^2}{4} - \frac{3y^2}{16} = 1$. Note to find the intersection points here is quite tricky – you have a quartic and you need to use the quadratic formula with x^2 as the variable. Alternatively you can eliminate x^2 and use the quadratic formula with y as the variable. Either way, you find the use of the quadratic formula gives a negative square root; therefore, there are no intersection points.
(c) The line has equation $y = 2x + 1$ and the hyperbola has equation $\frac{x^2}{4} - \frac{5y^2}{4} = 1$. There are no intersection points.
(d) The line has equation $y = \frac{-x}{2} + \frac{5}{2}$ and the ellipse has equation $\frac{x^2}{9} + \frac{5y^2}{36} = 1$. The intersection points are $(\frac{-1}{3}, \frac{8}{3})$ and $(\frac{19}{7}, \frac{8}{7})$.

Problem Set 3: Functions

Key Skills: Domains of Functions, Types of Functions, Properties of Functions

Practice Problems

Domains

Find the domain for each of the following functions. If any values of x are excluded, explain why.

a) $f(x) = \sqrt{x}$ *b)* $f(x) = \sqrt[3]{x}$ *c)* $f(x) = \sqrt{x^2}$ *d)* $f(x) = \sqrt{2x+2}$

e) $f(x) = \dfrac{1}{x}$ *f)* $f(x) = \dfrac{5}{x^2}$ *g)* $f(x) = \dfrac{1+x}{1-x}$ *h)* $f(x) = \dfrac{1+x}{1-x^2}$

i) $f(x) = \dfrac{e^x}{(2x-5)(x+2)}$ *j)* $f(x) = \dfrac{1}{e^x}$ *k)* $f(x) = \dfrac{1}{e^x - 1}$ *l)* $f(x) = \ln(x)$

m) $f(x) = \ln\left(\dfrac{1}{x}\right)$ *n)* $f(x) = \ln\left(\dfrac{1}{x^2}\right)$ *o)* $f(x) = \ln(x^2)$ *p)* $f(x) = \ln(\sin x)$

Properties of Functions

For each of these functions, determine if it is even, odd, or neither.
Hint: Write down $f(x)$ and $f(-x)$, and graph both of them.

a) $f(x) = 5$ *b)* $f(x) = \ln(x+2)$ *c)* $f(x) = e^{x^2}$ *d)* $f(x) = (e^x)^2$

e) $f(x) = \dfrac{1}{x}$ *f)* $f(x) = 5x^6$ *g)* $f(x) = \dfrac{5x^6}{3x^3}$ *h)* $f(x) = \dfrac{4}{x^2+2}$

Answers

Answers: Domains

$a)\ \ x \geq 0$ $\quad b)\ \ \text{all } \mathbb{R}$ $\quad c)\ \ \text{all } \mathbb{R}$ $\quad d)\ \ x \geq -1$

$e)\ \ x \neq 0$ $\quad f)\ \ x \neq 0$ $\quad g)\ \ x \neq 1$ $\quad h)\ \ x \neq \pm 1$

$i)\ \ x \neq \dfrac{5}{2} \text{ and } x \neq -2$ $\quad j)\ \ \text{all } \mathbb{R}$ $\quad k)\ \ x \neq 0$ $\quad l)\ \ x > 0$

$m)\ \ x > 0$ $\quad n)\ \ x \neq 0$ $\quad o)\ \ x \neq 0$ $\quad p)\ \ \text{all } \mathbb{R}$

Answers: Properties of Functions

a) even. b) neither. c) even. d) neither. e) odd. f) even. g) odd. h) neither.

Problem Set 4: Operations on Functions

Key Skills: Properties of Functions, Inversion, Composition

Practice Problems

Invert the following functions. To do so, give a reason for the invertibility of the function, restricting the domain if necessary. State the domain and range of the function, as well as the domain of the inverse. (You don't have to calculate the range of c), which is $(0, 1)$. You may assume that d), e) and f) are monotonic, since monotonicity is difficult to establish with our current tools.)

Demonstrate that you have the correct inverse by finding $f \circ f^{-1}(x)$ and $f^{-1} \circ f(x)$ on the appropriate domains.

$a)$ $f(x) = \dfrac{1}{e^x}$ $\quad b)$ $f(x) = \sqrt{x^2+1}$ $\quad c)$ $f(x) = \dfrac{1}{1+\frac{1}{(e^x)^2}}$

$d)$ $f(x) = \dfrac{4x-1}{2x+3}$ $\quad e)$ $f(x) = \dfrac{ax+b}{cx+d}$ $\quad f)$ $f(x) = \dfrac{e^x}{1+2e^x}$

$g)$ $f(x) = \ln\left(\dfrac{x}{x+1}\right)$ $\quad h)$ $f(x) = \dfrac{3e^x}{1-2e^x}$

Answers

(a) This function is defined on all of $\mathbb{R}$, and its range is $(0, \infty)$. The range is calculated by inspection: the exponential e^x is always positive, so its reciprocal is as well. e^x can be arbitrarily large or arbitrarily close to zero, so its reciprocal can realize any positive real number. f is a decreasing function, so it is invertible on its domain. Its inverse is defined on $(0, \infty)$ and has the form

$$f^{-1}(x) = -\ln(x)$$

(b) This function is defined on all of $\mathbb{R}$ and its range is $[1, \infty)$. The range is calculated by observing that $x^2 \geq 0$, so the term in the square root is ≥ 1. f a decreasing function on $(-\infty, 0)$ and an increasing functionon $(0, \infty)$, so we restrict its domain to $[0, \infty)$ to invert it. The range remains the same. Its inverse is defined on $[1, \infty)$ and has the form

$$f^{-1}(x) = \sqrt{x^2 - 1}$$

Note that by rearranging $y = \sqrt{x^2 + 1}$ into standard conic form, you can show that $f(x)$ is the top half of a hyperbola.

(c) This function is defined on all of $\mathbb{R}$ and its range is $(0, 1)$ as given. The function is increasing, so it is invertible. Its inverse is defined on $(0, 1)$ and has the form

$$f^{-1}(x) \;\; \ln \sqrt{\frac{1}{\frac{1}{x} - 1}}$$

(d) This function is defined on all $\mathbb{R}$ except $\frac{-3}{2}$. Its range is all $\mathbb{R}$ except 2. The range is difficult to calculate directly, but can be infered by looking at extreme values. Near the undefined value, the function gets arbitrarily large. For large x, the function gets arbitrarily close to 2. f is an increasing function that doesn't have conflicting values on either side of its asymptote, so it is invertible. The inverse is defined on all $\mathbb{R}$ except 2 and has the form

$$f^{-1}(x) = \frac{3x + 1}{4 - 2x}$$

(e) This function is the generalization of the previous function, with similar properties. It is monotonic, hence invertible, if $ad - bc \neq 0$. In that case, the domain is all $\mathbb{R}$ except $\frac{-d}{c}$ and the range is all $\mathbb{R}$ except $\frac{a}{c}$. The inverse is defined on all $\mathbb{R}$ except a/c and has the form

$$f^{-1}(x) = \frac{-dx + b}{cx - a}$$

Note this is an interesting case since all function of this form (with the condition $ad - bc \neq 0$) have inverses of essentially the same form. These function are called Möbius transformation; they are important in several areas of mathematics, particularly in complex analysis.

(f) The domain is all $\mathbb{R}$. The range is $\left(0, \frac{1}{2}\right)$. The range is difficult to calculate, but can be seen by looking at extreme values. For large negative x, the function is arbitrarily close to 0. For large x the function is arbitrarily close to $\frac{1}{2}$. For the rest of the function, it remains between those two values. The function is monotonic, hence invertible. The inverse is defined on $\left(0, \frac{1}{2}\right)$ and has the form

$$f^{-1}(x) = \ln\left(\frac{x}{1 - 2x}\right)$$

(g) When calculating the domain, remember that a fraction is positive when the numerator and denominator have the same sign, i.e. both positive or both negative; check both cases separately. The domain is all $\mathbb{R}$ except $[-1, 0]$. The range can again be seen by looking at extreme values, and at the behaviour near the boundaries. The function is monotonic, hence invertible. The inverse is defined for all $\mathbb{R}$ except 0, and has the form

$$f^{-1}(x) = \frac{e^x}{1 - e^x}$$

(h) The domain of f is all real except $\ln \frac{1}{2}$. The domain of f^{-1} is the union of the invervals $(-\infty, 0)$ and $\left(\frac{3}{2}, \infty\right)$. The inverse has the form

$$f^{-1}(x) = \ln\left(\frac{x}{2x+3}\right)$$

Problem Set 9: Limits

Key skills: Limit calculation

See Problem Set 11 for more advanced practice problems, with answers.

Practice Problems

a) $\lim\limits_{x\to -3} \left(\frac{\sin(\pi x)\cos(\pi x)}{x^2-4} + 2e^{\frac{x+3}{x}}\right)\left(x^2-x+1\right)$

b) $\lim\limits_{x\to 4} \frac{1}{1+\frac{1}{1+\frac{1}{x}}}$

c) $\lim\limits_{x\to 0} \frac{\tan^2 x}{x} + x^2\cot^2 x$

d) $\lim\limits_{x\to 0} \frac{x^3-2x^2+x}{\tan x}$

e) $\lim\limits_{x\to 2} \frac{x^2-4}{x-2}$

f) $\lim\limits_{x\to 0} \frac{4x-x^3}{3x+x^2}$

g) $\lim\limits_{x\to 0} \frac{(1+x)^2-1}{x}$

h) $\lim\limits_{x\to 1} \frac{x^4-1}{x^2+2x+3}$

i) $\lim\limits_{x\to 0^+} (1+x)^{\frac{3}{x}}$

j) $\lim\limits_{x\to 0} \frac{\sin^2 x-\tan^2 x}{x^2}$

k) $\lim\limits_{x\to 0} \frac{\sin^2 x-\tan^2 x}{x^2}$

Answers

Remember to start by checking if the limit is indeterminate, e.g. $\frac{0}{0}$ or $\frac{\infty}{\infty}$ etc.

$$a)\ \lim_{x\to -3}\left(\frac{\sin(\pi x)\cos(\pi x)}{x^2-4}+2e^{\frac{x+3}{x}}\right)\big(x^2-x+1\big)=26 \qquad b)\ \lim_{x\to 4}\frac{1}{1+\cfrac{1}{1+\cfrac{1}{x}}}=\frac{5}{9}$$

$$c)\ \lim_{x\to 0}\frac{\tan^2 x}{x}+x^2\cot^2 x=2 \qquad d)\ \lim_{x\to 0}\frac{x^3-2x^2+x}{\tan x}=1$$

$$e)\ \lim_{x\to 2}\frac{x^2-4}{x-2}=4 \qquad f)\ \lim_{x\to 0}\frac{4x-x^3}{3x+x^2}=\frac{4}{3} \qquad g)\ \lim_{x\to 0}\frac{(1+x)^2-1}{x}=2$$

$$h)\ \lim_{x\to 1}\frac{x^4-1}{x^2+2x+3}=0 \qquad i)\ \lim_{x\to 0^+}(1+x)^{\frac{3}{x}}=e^3 \qquad j)\ \lim_{x\to 0}\frac{\sin^2 x-\tan^2 x}{x^2}=0$$

$$k)\ \lim_{x\to 0}\frac{\sin^2 x-\tan^2 x}{x^2}=0$$

Questions a), b), d), and h) are solved by simple substitution. Questions c), j), i), and k) are solved by algebraic simplification to known limits. Questions e) $-$ g) are solved by algebraic manipulation and factoring.

Problem Set 11: Limits at Infinity

Includes practice problems for Lectures 10 and 11.

Key skills: Horizontal asymptotes; Asymptotic analysis

See last page for more advanced practice problems, with answers.

Practice Problems

a) $\lim_{x\to\infty} \frac{\sin x}{x^2 \cos x}$ *b)* $\lim_{x\to-\infty} \frac{\sin x}{x^2 \cos x}$

c) $\lim_{x\to\infty} \frac{\cos 3x}{x^2+2}$ *d)* $\lim_{x\to-\infty} \frac{\cos 3x}{x^2+2}$

e) $\lim_{x\to\infty} \frac{3x^3+2x+10}{9x^3-4x^2-100}$ *f)* $\lim_{x\to-\infty} \frac{3x^3+2x+10}{9x^3-4x^2-100}$

g) $\lim_{x\to\infty} \frac{5x^5-27x+2}{20x^4+20x+7}$ *h)* $\lim_{x\to-\infty} \frac{5x^5-27x+2}{20x^4+20x+7}$

i) $\lim_{x\to\infty} \frac{1}{x} - \frac{1}{x^2}$ *j)* $\lim_{x\to-\infty} \frac{1}{x} - \frac{1}{x^2}$

k) $\lim_{x\to\infty} \frac{e^{2x}x^3}{e^x x^6}$ *l)* $\lim_{x\to-\infty} \frac{e^{2x}x^3}{e^x x^6}$

m) $\lim_{x\to\infty} \frac{e^{x^2}x^3}{e^x x^6}$ *n)* $\lim_{x\to-\infty} \frac{e^{x^2}x^3}{e^x x^6}$

o) $\lim_{x\to\infty} \frac{(\ln|x|)^6}{x}$ *p)* $\lim_{x\to-\infty} \frac{(\ln|x|)^6}{x}$

q) $\lim_{x\to\infty} \frac{\sin^2 x}{x} - \frac{\cos^2 x}{x}$ *r)* $\lim_{x\to-\infty} \frac{\sin^2 x}{x} - \frac{\cos^2 x}{x}$

s) $\lim_{x\to\infty} \frac{\sqrt[3]{x^9+\sin^2 x}}{x^4+12}$ *t)* $\lim_{x\to-\infty} \frac{\sqrt[3]{x^9+\sin^2 x}}{x^4+12}$

Answers

a) $\lim_{x\to\infty} \frac{\sin x}{x^2 \cos x}$ DNE *b)* $\lim_{x\to-\infty} \frac{\sin x}{x^2 \cos x}$ DNE

c) $\lim_{x\to\infty} \frac{\cos 3x}{x^2+2} = 0$ *d)* $\lim_{x\to-\infty} \frac{\cos 3x}{x^2+2} = 0$

e) $\lim_{x\to\infty} \frac{3x^3+2x+10}{9x^3-4x^2-100} = \frac{1}{3}$ *f)* $\lim_{x\to-\infty} \frac{3x^3+2x+10}{9x^3-4x^2-100} = \frac{1}{3}$

g) $\lim_{x\to\infty} \frac{5x^5-27x+2}{20x^4+20x+7} = \infty$ *h)* $\lim_{x\to-\infty} \frac{5x^5-27x+2}{20x^4+20x+7} = -\infty$

i) $\lim_{x\to\infty} \frac{1}{x} - \frac{1}{x^2} = 0$ *j)* $\lim_{x\to-\infty} \frac{1}{x} - \frac{1}{x^2} = 0$

k) $\lim_{x\to\infty} \frac{e^{2x}x^3}{e^x x^6} = \infty$ *l)* $\lim_{x\to-\infty} \frac{e^{2x}x^3}{e^x x^6} = 0$

m) $\lim_{x\to\infty} \frac{e^{x^2}x^3}{e^x x^6} = \infty$ *n)* $\lim_{x\to-\infty} \frac{e^{x^2}x^3}{e^x x^6} = -\infty$

o) $\lim_{x\to\infty} \frac{(\ln|x|)^6}{x} = 0$ *p)* $\lim_{x\to-\infty} \frac{(\ln|x|)^6}{x} = 0$

q) $\lim_{x\to\infty} \frac{\sin^2 x}{x} - \frac{\cos^2 x}{x} = 0$ *r)* $\lim_{x\to-\infty} \frac{\sin^2 x}{x} - \frac{\cos^2 x}{x} = 0$

s) $\lim_{x\to\infty} \frac{\sqrt[3]{x^9+\sin^2 x}}{x^4+12} = 0$ *t)* $\lim_{x\to-\infty} \frac{\sqrt[3]{x^9+\sin^2 x}}{x^4+12} = 0$

Advanced Practice

Some more challenging problems:

$$\lim_{x\to\infty} \sqrt{x^2+2} - \sqrt{x^2-4} = 0$$

$$\lim_{x\to\infty} \sqrt{x^2+2} - \sqrt{x^3+2} = -\infty$$

$$\lim_{x\to\infty} \frac{(x+1)^{3x}\sqrt{x}}{\sqrt{x+1}x^{3x}} = e^3$$

$$\lim_{x\to 0} \frac{\sin 2x}{x^2 \cos x} = \text{DNE}$$

$$\lim_{x\to\infty} \frac{\sin 2x}{x^2 \cos x} = 0$$

$$\lim_{x\to 0} \frac{\tan x}{x^2 + x} = 0$$

$$\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = 1$$

$$\lim_{x\to 3} \frac{x^3 + 6x - 7}{x^4 - 81} = \pm\infty$$

$$\lim_{x\to 3} \left(\frac{x^2+4x-21}{x^2-9}\right)^{3/2} = \sqrt{\frac{125}{27}}$$

$$\lim_{x\to 0} \frac{\frac{1}{3+x} - \frac{1}{3-x}}{x} = \frac{-2}{9}$$

$$\lim_{x\to 3} \frac{1}{x-3}\left(\frac{1}{\sqrt{x+1}} - \frac{1}{2}\right) = \frac{1}{16}$$

$$\lim_{x\to -4} \frac{\frac{1}{\sqrt{13+x}} - \frac{1}{3}}{x+4} = \frac{-1}{54}$$

$$\lim_{x\to 4} \frac{\sqrt{1+2x} - 3}{\sqrt{x} - 2} = \frac{4}{3}$$

$$\lim_{x\to 4} \frac{x^{3/2} - 8}{x - 4} = -3$$

Problem Set 12: Limits, Asymptotics, and Models

Key skills: Asymptotic analysis; Mathematical models

Practice Problems

For the following population models, determine the initial population value p at $t = 0$ and the long term behaviour of the population. If the population model has a long-term steady state, determine the final population value. Classify the growth: which are exponential, which are logistic, and which are some other type of growth?

Assume that t is measured in years and p is measured in thousands of individuals.

a) $p(t) = 100e^{\frac{1}{2}t}$

b) $p(t) = 20e^{-6t}$

c) $p(t) = \dfrac{1200e^{\frac{5}{9}t}}{88 + 12\left(e^{\frac{5}{9}t}\right)}$

d) $p(t) = \dfrac{50e^{\frac{1}{10}t}}{23 + 2\left(e^{\frac{1}{10}t}\right)}$

e) $p(t) = \dfrac{50e^{\frac{1}{10}t}}{23\cos(6\pi t) + 2\left(e^{\frac{1}{10}t}\right)}$

f) $p(t) = \dfrac{200\left(\frac{t}{3}\right)^2}{35 + 5\left(\frac{t}{3}\right)^2}$

g) $p(t) = 90\left(\dfrac{t}{7}\right)^2 + 10$

h) $p(t) = \dfrac{300}{\frac{t}{2} + 1}$

i) $p(t) = \dfrac{200\left(\frac{t}{6}\right)}{77e^{\frac{1}{6}t}}$

j) $p(t) = \dfrac{9000t^2 + 900t + 9}{450t^2 - 31t + 3}$

k) $p(t) = \dfrac{22t^3 + 396t^2 - 88t + 8}{511t^2 + 2t + 4}$

l) $p(t) = \dfrac{64t^2 + 128t}{27t^3 + 9t + 3}$

Answers

a) $p(0) = 100$. The population grows without limit. This is an exponential growth model.

b) $p(0) = 20$. The population eventually decreases to 0. This is an exponential decay model.

c) $p(0) = \frac{1200}{88+12} = 12$. The population eventually stabilizes at $\frac{1200}{12} = 100$. This is a logistic growth model.

d) $p(0) = \frac{50}{23+2} = 2$. The population eventually stabilizes at $\frac{50}{2} = 25$. This is a logistic growth model.

e) $p(0) = \frac{50}{23+2} = 2$. The population eventually stabilizes at $\frac{50}{2} = 25$. (As t grows the cosine term becomes insignificant asymptotically.) This is not any of the types of growth models listed.

f) $p(0) = 0$. The population eventually stabilizes at $\frac{200}{5} = 40$. This is not any of the types of growth models listed.

g) $p(0) = 10$. The population grows without limit. This is not any of the types of growth models listed.

h) $p(0) = 300$. The population eventually decreases to 0. This is not any of the types of growth models listed.

i) $p(0) = 0$. The population eventually decreases to 0. This is not any of the types of growth models listed.

j) $p(0) = 3$. The population eventually stabilizes at $\frac{9000}{450} = 20$. This is not any of the types of growth models listed.

k) $p(0) = 2$. The population grows without limit. This is not any of the types of growth models listed.

l) $p(0) = 0$. The population becomes positive and eventually decreases back to 0. This is not any of the types of growth models listed.

Problem Set 13: Continuity

Key skills: Continuity, Piecewise Functions

Practice Problems

For each of these functions, determine whether or not the function is continuous at the given x value.

$a)\ f(x) = x^2$ at $x = 0$ $\qquad b)\ f(x) = \dfrac{1}{x^2}$ at $x = 0$ $\qquad c)\ f(x) = \dfrac{x+2}{(x+2)(x+4)}$ at $x = -2$

For each of these piecewise functions, determine whether or not the function is continuous at its crossover point.

$$d)\ f(x) = \begin{cases} 2 & x \le 1 \\ 4 & x > 1 \end{cases} \qquad e)\ f(x) = \begin{cases} 2x & x < 1 \\ x^2 + 3x & x \ge 1 \end{cases} \qquad f)\ f(x) = \begin{cases} 3x + 2 & x < 2 \\ x^3 & x = 2 \\ 4x^2 - 8 & x > 2 \end{cases}$$

For each of these piecewise functions, find a value for a which makes the function continuous, or show that no such value exists.

$$g)\ f(x) = \begin{cases} x^2 + 3x - 1 & x \ge 2 \\ \frac{a}{x} & x < 2 \end{cases} \qquad h)\ f(x) = \begin{cases} x^2 + 3x - 1 & x \ge 2 \\ \frac{a}{x-2} & x < 2 \end{cases}$$

Answers

a) $\lim_{x\to 0^-} f(x) = 0$. $\lim_{x\to 0^+} f(x) = 0$. $f(0) = 0$. Continuous.

b) Discontinuous; $f(0)$ is not defined.

c) Discontinuous; $f(-2)$ is not defined. (Note that the $(x+2)$ factors cancel, but they can *only* be cancelled where $x + 2 \neq 0$.)

d) Discontinuous; $f(x)$ approaches (horizontally!) different values from each direction.

e) $\lim_{x\to 1^-} 2x = 2$. $\lim_{x\to 1^+} x^2 + 3x = 4$. Discontinuous.

f) $\lim_{x\to 2^-} 3x + 2 = 8$. $(2)^3 = 8$. $\lim_{x\to 2^+} 4x^2 - 8 = 8$. Continuous.

g) $\lim_{x\to 2^+} f(x) = 9$. $\lim_{x\to 2^-} f(x) = \frac{a}{2}$. Continuous if $a = 18$.

h) $\lim_{x\to 2^+} f(x) = 9$. $\lim_{x\to 2^-} f(x) = -\infty$. Not continuous for any a.

Problem Set 16: Derivatives

Key skills: Derivatives using Derivative Rules; Derivatives of inverse functions

Combines material from Lectures 14, 15, and 16.

Practice Problems

Derivative Rules

Do the following derivatives, explicitly labelling the rules you are using. (Power, Linearity, Product, Quotient, Chain). Simplify if you want, and if you see an obvious simplification, but don't worry about simplifying difficult expressions.

The derivative $\frac{d}{dx}e^{kx}$ (problem i) is particularly useful, and it's worth memorizing the answer once you find it. There are (at least) two different ways to do problem b.

$a)$ $\dfrac{d}{dx}x^2 \sin x \ln x$ $\qquad$ $b)$ $\dfrac{d}{dx}\dfrac{a^x}{b^x}$ $\qquad$ $c)$ $\dfrac{d}{dx}\cos(x^2+1)$ $\qquad$ $d)$ $\dfrac{d}{dx}\sin(\cos(\sin x)))$

$e)$ $\dfrac{d}{dx}\sqrt{1-\dfrac{1}{x^2+1}}$ $\qquad$ $f)$ $\dfrac{d}{dx}x^2e^x + x^3e^{x-1}$ $\qquad$ $g)$ $\dfrac{d}{dx}e^{\sin x+\cos x}$ $\qquad$ $h)$ $\dfrac{d}{dx}2^{x^4}$

$i)$ $\dfrac{d}{dx}e^{kx}$ $\qquad$ $j)$ $\dfrac{d}{dx}2^{x^2+x}+x^2$

Derivatives of Inverse Functions

Do the following derivatives using the rule for derivatives of inverse functions.

$k)$ $\dfrac{d}{dx}\sqrt{x}$ $\qquad$ $l)$ $\dfrac{d}{dx}\sqrt{x^2-1}$ $\qquad$ $m)$ $\dfrac{d}{dx}\ln(x^2)$

Answers

Derivative Rules

$a)\quad \dfrac{d}{dx}x^2\sin x\ln x = x^2\cos x\ln x + 2x\sin x\ln x + x\sin x \qquad b)\quad \dfrac{d}{dx}\dfrac{a^x}{b^x} = \dfrac{a^x(\ln a - \ln b)}{b^x}$

$c)\quad \dfrac{d}{dx}\cos(x^2+1) = -2x\sin(x^2+1) \qquad d)\quad \dfrac{d}{dx}\sin(\cos(\sin x)) = -\cos(\cos(\sin x)))\sin(\sin x)\cos x$

$e)\quad \dfrac{d}{dx}\sqrt{1-\dfrac{1}{x^2+1}} = \dfrac{1}{2\sqrt{1-\frac{1}{x^2+1}}}\dfrac{2x}{(x^2+1)^2} \qquad f)\quad \dfrac{d}{dx}x^2e^x+x^3e^{x-1} = 2xe^x+x^2e^x+3x^2e^{x-1}+x^3e^{x-1}$

$g)\quad \dfrac{d}{dx}e^{\sin x+\cos x} = (\cos x - \sin x)e^{\sin x+\cos x} \qquad h)\quad \dfrac{d}{dx}2^{x^4} = 2^{x^4}4x^3\ln 2$

$i)\quad \dfrac{d}{dx}e^{kx} = ke^{kx} \qquad j)\quad \dfrac{d}{dx}2^{x^2+x}+x^2 = 2^{x^2+x}\ln 2(2x+2)+2x$

Derivatives of Inverse Functions

$k)\quad \dfrac{d}{dx}\sqrt{x}$

Use $f(x) = x^2$ so $f^{-1}(x) = \sqrt{x}$.

$$\frac{d}{dx}\sqrt{x} = \left.\frac{1}{2u}\right|_{u=\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$l)\quad \dfrac{d}{dx}\sqrt{x^2-1}$

Use $f(x) = (x+1)^2$ so $f^{-1}(x) = \sqrt{x^2-1}$

$$\frac{d}{dx}\sqrt{x^2-1} = \left.\frac{1}{2(u+1)}\right|_{u=\sqrt{x+1}} = \frac{1}{2\sqrt{x+1}}$$

$m)\quad \dfrac{d}{dx}\ln(x^2)$

Use $f(x) = \sqrt{e^x}$ so $f^{-1}(x) = \ln x^2$

$$\frac{d}{dx}\ln(x^2) = \left.\frac{1}{\frac{e^u}{2\sqrt{e^u}}}\right|_{u=\ln x^2} = \frac{2\sqrt{e^{\ln x^2}}}{e^{\ln x^2}} = \frac{2x}{x^2} = \frac{2}{x}$$

(This last can be simplified a great deal by writing $\ln(x^2) = 2\ln x$, but it useful to do in this way to see how to use the inverse function derivative process.)

Problem Set 17: Implicit Derviatives

Key skills: Implicit Differentiation, Chain Rule, Tangent Lines

Practice Problems

For each of the following curves, find a general expression for the implicit derivative $\frac{dy}{dx}$, determine where the derivative is undefined, and calculate the tangent line at the given points.

It is strongly recommended that you first plot each curve, using Desmos or another curve-graphing program.

a) $x^2 + y^2 = 36$, tangent at $(0, 6)$, at $(3\sqrt{2}, 3\sqrt{2})$, and at $(-3\sqrt{2}, 3\sqrt{2})$.

b) $x^2 - y^2 = 36$, tangent at $(\sqrt{1000036}, 1000)$

c) $y^2 - x = 0$, tangent at $(1, 1)$, and at $(1, -1)$

d) $x^2 - y = 0$, tangent at $(1, 1)$

e) $y^2 + \cos x = 1$, tangent at $(\pi, \sqrt{2})$, and at $\left(\frac{\pi}{2}, -1\right)$

Answers

a) $\dfrac{dy}{dx} = -\dfrac{x}{y}$

$(0, 6)$: The slope is 0, so the tangent line is $y = 6$ (a constant).

$(3\sqrt{2},\ 3\sqrt{2})$: The slope is -1. To find the intercept: $(3\sqrt{2}) = -(3\sqrt{2}) + b$, so $b = 6\sqrt{2}$. Then the tangent line is $y = -x + 6\sqrt{2}$.

$(-3\sqrt{2},\ 3\sqrt{2})$: The slope is 1. To find the intercept: $(3\sqrt{2}) = (-3\sqrt{2}) + b$, so again $b = 6\sqrt{2}$. (You can see this from symmetry if you sketch these tangent lines on the curve.) Then the tangent line is $y = x + 6\sqrt{2}$

b) $\dfrac{dy}{dx} = \dfrac{x}{y}$

$(\sqrt{1000036},\ 1000)$: The slope is $\frac{\sqrt{1000036}}{1000} \approx 1.000018 \approx 1$. To a very good approximation, the tangent line is $y = x$.

c) $\dfrac{dy}{dx} = \dfrac{1}{2y}$

$(1, 1)$: The slope is $1/2$. To find the intercept: $(1) = \frac{1}{2}(1) + b$, so $b = 1/2$. Then the tangent line is $y = \frac{1}{2}x + \frac{1}{2}$.

$(1, -1)$: The slope is $-1/2$. To find the intercept: $(-1) = -\frac{1}{2}(1) + b$, so $b = -1/2$. Then the tangent line is $y = -\frac{1}{2}x - \frac{1}{2}$.

d) $\dfrac{dy}{dx} = 2x$

$(1, 1)$: The slope is 2. To find the intercept: $(1) = 2(1) + b$, so $b = -1$. Then the tangent line is $y = 2x - 1$.

e) $\dfrac{dy}{dx} = \dfrac{\sin x}{2y}$

$(\pi, \sqrt{2})$: The slope is 0, so the tangent line is $y = \sqrt{2}$ (a constant).

$\left(\frac{\pi}{2}, -1\right)$: The slope is $-\frac{1}{2}$. To find the intercept: $-1 = -\frac{1}{2}\left(\frac{\pi}{2}\right) + b$, so $b = \frac{\pi}{4} - 1$. Then the tangent line is $y = -\frac{1}{2}x + \frac{\pi}{4} - 1$.

Problem Set 18: Higher Derivatives

Key skills: Higher Derivatives (a.k.a. Multiple Derivatives)

Practice Problems

Watch for patterns in your solutions to some of these problems.

The patterns you find in problems c, g, h, i, and j are particularly interesting and useful.

$a)\dfrac{d^2}{dx^2}x^4-3x+1$ $\quad b)\dfrac{d^2}{dx^2}\sin(x^2)$ $\quad c)\dfrac{d^2}{dx^2}e^x$

$d)\dfrac{d^2}{dx^2}\cos(x^2-1)$ $\quad e)\dfrac{d^2}{dx^2}\dfrac{x+1}{x^2+1}$ $\quad f)\dfrac{d^2}{dx^2}\ln(1-x^2)$

$g)\dfrac{d^3}{dx^3}x^4-5x^3+6x+3$ $\quad h)\dfrac{d^5}{dx^5}x^4-5x^3+6x+3$

$i)\dfrac{d^4}{dx^4}\sin(4x)$ $\quad j)\dfrac{d^4}{dx^4}e^{6x+1}$ $\quad k)\dfrac{d^3}{dx^3}e^{2x}$

Answers

$$a)\frac{d^2}{dx^2}x^4 - 3x + 1 = \frac{d}{dx}4x^3 - 3 = 12x^2$$

$$b)\frac{d^2}{dx^2}\sin(x^2) = \frac{d}{dx}2x\cos(x^2) = 2\cos(x^2) - 4x^2\sin(x^2)$$

$$c)\frac{d^2}{dx^2}e^x = \frac{d}{dx}e^x = e^x$$

$$d)\frac{d^2}{dx^2}\cos(x^2-1) = \frac{d}{dx} - 2x\sin(x^2-1) = -2\sin(x^2-1) - 4x^2\cos(x^2-1)$$

$$\begin{aligned} e)\frac{d^2}{dx^2}\frac{x+1}{x^2+1} &= \frac{d}{dx}\frac{x^2+1-(x+1)(2x)}{(x^2+1)^2} = \frac{d}{dx}\frac{-x^2-2x+1}{(x^2+1)^2} \\ &= \frac{(-2x-2)(x^2+1)^2 - (-x^2-2x+1)2(x^2+1)(2x)}{(x^2+1)^4} \\ &== \frac{(-2x-2)(x^2+1) - (-x^2-2x+1)(4x)}{(x^2+1)^3} \\ &= \frac{-2x^3-2x^2-2x-2-(-4x^3-8x^2+4x)}{(x^2+1)^3} = \frac{2x^3+6x^2-6x-2}{(x^2+1)^3} \end{aligned}$$

$$f)\frac{d^2}{dx^2}\ln(1-x^2) = \frac{d}{dx}\frac{-2x}{1-x^2} = \frac{-2(1-x^2)-(-2x)(-2x)}{(1-x^2)^2} = \frac{-2x^2-2}{(1-x^2)^2}$$

$$g)\frac{d^3}{dx^3}x^4 - 5x^3 + 6x + 3 = \frac{d^2}{dx^2}4x^3 - 15x^2 + 6 = \frac{d}{dx}12x^2 + 30x = 24x + 30$$

$$h)\frac{d^5}{dx^5}x^4 - 5x^3 + 6x + 3 = \frac{d^4}{dx^4}4x^3 - 15x^2 + 6 = \frac{d^3}{dx^3}12x^2 + 30x = \frac{d^2}{dx^2}24x + 30 = \frac{d}{dx}24 = 0$$

$$i)\,\frac{d^4}{dx^4}\sin(4x) = \frac{d^3}{dx^3}4\cos(4x) = \frac{d^2}{dx^2}(-16\sin(4x)) = \frac{d}{dx}(-64\cos(4x)) = 256\sin(4x)$$

$$j)\,\frac{d^4}{dx^4}e^{6x+1} = \frac{d^3}{dx^3}6e^{6x+1} = \frac{d^2}{dx^2}36e^{6x+1} = \frac{d}{dx}216e^{6x+1} = 1296e^{6x+1}$$

$$\text{or}\quad \frac{d^4}{dx^4}e^{6x+1} = \frac{d^3}{dx^3}6e^{6x+1} = \frac{d^2}{dx^2}6^2e^{6x+1} = \frac{d}{dx}6^3e^{6x+1} = 6^4e^{6x+1}$$

$$k)\,\frac{d^3}{dx^3}e^{2x} = \frac{d^2}{dx^2}2e^{2x} = \frac{d}{dx}4e^{2x} = 8e^{2x}$$

Problem Set 19: Linear Approximation

Key skills: Tangent Lines as approximations of functions

Practice Problems

Write the linear approximation (tangent line) of the given function at the given points. Use it to calculate nearby values. If you have a computer to give approximate values, compare the results from your approximation to what your computer gives.

(a) The function $f(x) = \sin x$ near the points $(0, 0)$ and $(\pi, 0)$.

(b) The function $f(x) = \cos x$ near the point $(0, 1)$ and $(2\pi, 1)$.

(c) The function $f(x) = \tan x$ near the point $(0, 0)$.

(d) The function $f(x) = x^2$ near the points $(0, 0)$ and $(1, 1)$.

(e) The function $f(x) = x^3$ near the points $(0, 0)$ and $(1, 1)$.

(f) The function $f(x) = (x - 5)^2$ near the points $(5, 0)$ and $(0, 25)$.

(g) The function $f(x) = e^x$ near the point $(0, 1)$.

(h) The function $f(x) = \frac{x+1}{2x^2-1}$ near the points $(0, -1)$ and $(1, 2)$.

Answers

(a) $\frac{d}{dx}\sin x = \cos x$
Near $(0,0)$: $f(x) \approx x$
Near $(\pi, 0)$: $f(x) \approx -x + \pi$

(b) $\frac{d}{dx}\cos x = -\sin x$
Near $(0,1)$: $f(x) \approx 1$
Near $(2\pi, 1)$: $f(x) \approx 1$

(c) $\frac{d}{dx}\tan x = \sec^2 x$
Near $(0,0)$: $f(x) \approx x$

(d) $\frac{d}{dx}x^2 = 2x$
Near $(0,0)$: $f(x) \approx 0$
Near $(1,1)$: $f(x) \approx 2x - 1$

(e) $\frac{d}{dx}x^3 = 3x^2$
Near $(0,0)$: $f(x) \approx 0$
Near $(1,1)$: $f(x) \approx 3x - 2$

(f) $\frac{d}{dx}(x-5)^2 = 2x - 10$
Near $(5,0)$: $f(x) \approx 0$
Near $(0,25)$: $f(x) \approx -10x + 25$

(g) $\frac{d}{dx}e^x = e^x$
Near $(0,1)$: $f(x) \approx x + 1$

(h) $\frac{d}{dx}\frac{x+1}{2x^2-1} = -\frac{2x^2+4x+1}{(2x^2-1)^2}$
Near $(0,-1)$: $f(x) \approx -x - 1$
Near $(1,2)$: $f(x) \approx -7x + 9$

Problem Set 20: Sigma Notation

Key skills: Sigma notation for sums, manipulating sums, combining sums, manipulating indices

Practice Problems

1) Write out each of the following sums in full. (Questions a, b, c, h, and i demonstrate how shifting indices works. Question f shows why you can factor constants out of a sum.)

$a)\ \sum_{k=1}^{4} k \qquad b)\ \sum_{k=2}^{5}(k-1) \qquad c)\ \sum_{k=3}^{6}(k-2) \qquad d)\ \sum_{k=2}^{5} k \qquad e)\ \sum_{j=1}^{4} j \qquad f)\ \sum_{k=1}^{4} 2k$

$g)\ \sum_{k=1}^{4} 1 \qquad h)\ \sum_{k=0}^{2} k^2 \qquad i)\ \sum_{k=1}^{3}(k-1)^2 \qquad j)\ \sum_{k=1}^{3}(k^2-1) \qquad k)\ \sum_{k=1}^{3}(k^3-4k)$

2) For each of these sums, take out the first three terms and use sigma notation for the rest.

$a)\ \sum_{k=1}^{10} \frac{k^2}{2} \qquad b)\ \sum_{k=1}^{5}(2k-1) \qquad c)\ \sum_{k=1}^{8}(2k^2+k) \qquad d)\ \sum_{k=3}^{9} k$

3) Combine each pair of sums into a single sigma-notation expression, shifting indices where necessary. *(Tip: When shifting indices of a sum, check it by writing out the first two or three terms; the original and "shifted" versions should work out the same.)*

$a)\ \sum_{k=1}^{12} 3k + \sum_{k=1}^{12} 5k^2 \qquad b)\ \sum_{k=1}^{7} 2k + \sum_{k=3}^{9} 3k \qquad c)\ \sum_{k=2}^{4} 4k^2 + \sum_{k=2}^{4} 2k^2 \qquad d)\ \sum_{k=3}^{25}(k^2+2k) + \sum_{k=1}^{23} k+2$

4) Demonstrate the following formulas for the given values of n.

$a)\ \sum_{k=1}^{n} 1 = n \quad \text{for } n=5$

$b)\ \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \text{for } n=6 \text{ and } n=7$

$c)\ \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{for } n=4$

$d)\ \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2 \quad \text{for } n=3$

Answers

Part 1:

a) $\displaystyle\sum_{k=1}^{4} k = 1 + 2 + 3 + 4$

b) $\displaystyle\sum_{k=2}^{5}(k-1) = (2-1) + (3-1) + (4-1) + (5-1) = 1 + 2 + 3 + 4$

c) $\displaystyle\sum_{k=3}^{6}(k-2) = (3-2) + (4-2) + (5-2) + (6-2) = 1 + 2 + 3 + 4$

d) $\displaystyle\sum_{k=2}^{5} k = 2 + 3 + 4 + 5$

e) $\displaystyle\sum_{j=1}^{4} j = 1 + 2 + 3 + 4$

f) $\displaystyle\sum_{k=1}^{4} 2k = 2(1) + 2(2) + 2(3) + 2(4) = 2 + 4 + 6 + 8$

g) $\displaystyle\sum_{k=1}^{4} 1 = 1 + 1 + 1 + 1$

h) $\displaystyle\sum_{k=0}^{2} k^2 = 0^2 + 1^2 + 2^2 = 0 + 1 + 4$

i) $\displaystyle\sum_{k=1}^{3}(k-1)^2 = (1-1)^2 + (2-1)^2 + (3-1)^2 = 0^2 + 1^2 + 2^2 = 0 + 1 + 4$

j) $\displaystyle\sum_{k=1}^{3}(k^2-1) = ((1)^2-1) + ((2)^2-1) + ((3)^2-1) = (1-1) + (4-1) + (9-1) = 0 + 3 + 8$

k) $\displaystyle\sum_{k=1}^{3}(k^3-4k) = (1^3-4(1)) + (2^3-4(2)) + (3^3-4(3)) = (1-4) + (8-8) + (27-12) = 3 + 0 + 15$

Part 2:

a) $$\sum_{k=1}^{10} \frac{k^2}{2} = \frac{1}{2} + \frac{4}{2} + \frac{9}{2} + \sum_{k=4}^{10} \frac{k^2}{3}$$

b) $$\sum_{k=1}^{5}(2k-1) = 1 + 3 + 5 + \sum_{k=4}^{5}(2k-1)$$

c) $$\sum_{k=1}^{8}(2k^2+k) = 3 + 10 + 21 + \sum_{k=4}^{8}(2k^2+k)$$

d) $$\sum_{k=3}^{9} k = 3 + 4 + 5 + \sum_{k=6}^{9} k$$

Part 3:

a) $$\sum_{k=1}^{12} 3k + \sum_{k=1}^{12} 5k^2 = \sum_{k=1}^{12}(3k+5k^2)$$

b) $$\sum_{k=1}^{7} 2k + \sum_{k=3}^{9} 3k = \sum_{k=3}^{9} 2(k-1) + \sum_{k=3}^{9} 3k = \sum_{k=3}^{9} 5k - 2$$

or: $$\sum_{k=1}^{7} 2k + \sum_{k=3}^{9} 3k = \sum_{k=1}^{7} 2k + \sum_{k=1}^{7} 3(k+2) = \sum_{k=1}^{7} 5k + 2$$

(Convince yourself these are the same result by writing out the first few terms!)

c) $$\sum_{k=2}^{4} 4k^2 + \sum_{k=2}^{4} 2k^2 = \sum_{k=2}^{4} 6k^2$$

d) $$\sum_{k=3}^{25}(k^2+2k) + \sum_{k=1}^{23} k + 2 = \sum_{k=3}^{25}(k^2+2k) + \sum_{k=3}^{25} k = \sum_{k=3}^{25}(k^2+3k)$$

(You could also shift the first sum instead, but this is simpler.)

Part 4:

a) LHS: $\sum_{k=1}^{5} 1 = 1+1+1+1+1 = 5$

RHS: $n = 5 \quad \checkmark$

b) LHS: $\sum_{k=1}^{6} k = 1+2+3+4+5+6 = 21$

RHS: $\dfrac{6(6+1)}{2} = 21 \quad \checkmark$

LHS: $\sum_{k=1}^{7} k = 1+2+3+4+5+6+7 = 28$

RHS: $\dfrac{7(7+1)}{2} = 28 \quad \checkmark$

c) LHS: $\sum_{k=1}^{4} k^2 = 1+4+9+16 = 30$

RHS: $\dfrac{4(4+1)(2(4)+1)}{6} = 30 \quad \checkmark$

d) LHS: $\sum_{k=1}^{3} k^3 = 1+8+27 = 36$

RHS: $\left(\dfrac{3(3+1)}{2}\right)^2 = 36 \quad \checkmark$

Problem Set 22: Integration using the Fundamental Theorem of Calculus

Key skills: Integration, Fundamental Theorem of Calculus, Antiderivatives

Practice Problems

Calculate the following integrals by directly finding an anti-derivative or using tables. For the indefinite integrals, check your answer by differentiation.

a) $\displaystyle\int x^3\,dx$ *b)* $\displaystyle\int 3^x\,dx$ *c)* $\displaystyle\int (x^2+x+1)\,dx$

d) $\displaystyle\int \sin(2x)\,dx$ *e)* $\displaystyle\int \frac{1}{x^2}\,dx$ *f)* $\displaystyle\int 6\sec^2 x\,dx$

g) $\displaystyle\int_1^2 \left(\frac{1}{x^2}+x\right)\,dx$ *h)* $\displaystyle\int_1^7 e^x\,dx$

i) $\displaystyle\int_{-\pi}^{\pi} \cos x\,dx$ *j)* $\displaystyle\int_0^{\pi} (\sin x-\cos x)\,dx$ *k)* $\displaystyle\int_{-1}^{1} (x-1)^2\,dx$

Answers

$a)\ \int x^3\,dx = \frac{x^4}{4} + C$ $\qquad c)\ \int 3^x\,dx = \frac{3^x}{\ln 3} + C$ $\qquad b)\ \int (x^2 + x + 1)\,dx = \frac{x^3}{3} + \frac{x^2}{2} + x + C$

$d)\ \int \sin(2x)\,dx = \frac{-\cos(2x)}{2} + C$ $\qquad e)\ \int \frac{1}{x^2}\,dx = \frac{-1}{x} + C$ $\qquad f)\ \int 6\sec^2 x\,dx = 6\tan x + C$

$g)\ \int_1^2 \left(\frac{1}{x^2} + x\right)\,dx = 2$ $\qquad h)\ \int_1^7 e^x\,dx = e^7 - e$

$i)\ \int_{-\pi}^{\pi} \cos x\,dx = 0$ $\qquad j)\ \int_0^{\pi} (\sin x - \cos x)\,dx = 2$ $\qquad k)\ \int_{-1}^{1} (x-1)^2\,dx = \frac{8}{3}$

Problem Set 23: Integration using Substitution

Key skills: Substitution

Practice Problems

Do the following integrals by substitution. Clearly label your chosen substitution.

a) $\displaystyle\int \frac{2x^2}{\sqrt{1-4x^3}}\,dx$ *b)* $\displaystyle\int 2x(x^2-2)^{80}\,dx$ *c)* $\displaystyle\int x^7(\cos x^8)\,dx$

d) $\displaystyle\int \frac{(\sqrt{x}+1)^4}{2\sqrt{x}}\,dx$ *e)* $\displaystyle\int (x^3+x)^7(3x^2+1)\,dx$ *f)* $\displaystyle\int \cos x \sin^4 x\,dx$

g) $\displaystyle\int_0^4 \frac{4x}{\sqrt{4+x^2}}\,dx$ *h)* $\displaystyle\int_0^{\pi/2} \cos x \sin^2 x\,dx$ *i)* $\displaystyle\int_0^3 \frac{x^2+1}{\sqrt{x^3+3x+4}}\,dx$

j) $\displaystyle\int_1^{e^2} \frac{\ln x}{x}\,dx$

Answers

$$a)\int \frac{2x}{\sqrt{1-4x^3}}\,dx = \frac{-\sqrt{1-4x^3}}{3} \qquad u = 1-4x^3 \qquad du = -12x^2\,dx$$

$$b)\int 2x(x^2-2)^{80}\,dx = \frac{(x^2-2)^{81}}{81} \qquad u = x^2-2 \qquad du = 2x\,dx$$

$$c)\int x^7(\cos x^8)\,dx\,dx = \frac{-\sin x^8}{8} \qquad u = x^8 \qquad du = 8x^7\,dx$$

$$d)\int \frac{(\sqrt{x}+1)^4}{2\sqrt{x}}\,dx = \frac{(\sqrt{x}+1)^5}{5} \qquad u = \sqrt{x}+1 \qquad du = \frac{1}{2\sqrt{x}}\,dx$$

$$e)\int (x^3+x)^7(3x^2+1)\,dx = \frac{(x^3+x)^8}{8} \qquad u = x^3+x \qquad du = (3x^2+1)\,dx$$

$$f)\int \cos x \sin^4 x\,dx = \frac{\sin^5 x}{5} \qquad u = \sin x \qquad du = \cos x\,dx$$

$$g)\int_0^4 \frac{4x}{\sqrt{4+x^2}}\,dx = \frac{32}{2}(5\sqrt{5}-1) \qquad u = 4+x^2 \qquad du = 2x\,dx$$

$$h)\int_0^{\pi/2} \cos x \sin^2 x\,dx = \frac{1}{3} \qquad u = \sin x \qquad du = \cos x\,dx$$

$$i)\int_0^3 \frac{x^2+1}{\sqrt{x^3+3x+4}}\,dx = \frac{2\sqrt{40}-4}{3} \qquad u = x^3+3x+4 \qquad du = (3x^2+3)\,dx$$

$$j)\int_1^{e^2} \frac{\ln x}{x}\,dx = 2 \qquad u = \ln x \qquad du = \frac{1}{x}\,dx$$

Problem Set 24: Separable Differential Equations

Key skills: Separable Differential Equations.

Practice Problems

Solve the following separable initial value problems.

a) $\dfrac{df}{dx} = f \qquad f(0) = 50$

b) $\dfrac{df}{dx} = \dfrac{3\sqrt[3]{f}}{2} \qquad f(0) = 0$

c) $\dfrac{df}{dx} = \dfrac{2}{f} \qquad f(2) = 3$

d) $\dfrac{df}{dx} = x^2 f^2 \qquad f(0) = 7$

e) $\dfrac{df}{dx} = \dfrac{1}{xf} \qquad f(1) = 4$

Answers

a) $f(x) = 50e^x$

b) $f(x) = \sqrt{x^3}$

c) $f(x) = \sqrt{4x+1}$

d) $f(x) = \dfrac{-3}{x^3 - \frac{3}{7}}$

e) $f(x) = \sqrt{2\ln|x| + 16}$

Problem Set 25: L'Hôpital's Rule

Key skills: Derivatives, Limits

Practice Problems

Applying L'Hôpital's rule

Use L'Hôpital's rule to calculate the following limits.

$a)\ \lim_{x\to 0}\frac{\sin x}{x}$ $b)\ \lim_{x\to 0}\frac{x}{\sin x}$ $c)\ \lim_{x\to 0}\frac{\tan x}{x}$ $d)\ \lim_{x\to 1}\frac{x-1}{x^2-1}$

$e)\ \lim_{x\to\infty}\frac{\ln x}{x}$ $f)\ \lim_{x\to\infty}\frac{x+e^x}{xe^x}$ $g)\ \lim_{x\to\infty}\frac{\frac{1}{x^4}-\frac{1}{x^2}}{\ln x}$ $h)\ \lim_{x\to\infty}\frac{\ln(x^2)}{(\ln x)^2}$

Forbidden questions

Explain why you *cannot* apply L'Hôpital's rule for any of the following limits.

$a)\ \lim_{x\to\pi}\frac{\sin x}{x}$ $b)\ \lim_{x\to 0}\frac{\cos x}{x}$ $c)\ \lim_{x\to -1}\frac{x-1}{x^2-1}$

$d)\ \lim_{x\to 4}\frac{x^2-16}{e^x}$ $e)\ \lim_{x\to 4}\frac{x^2}{e^x}$ $f)\ \lim_{x\to\infty}\frac{e^{-x}}{x}$

Answers

Answers to: Applying L'Hôpital's rule

a) $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$

b) $\lim_{x\to 0} \frac{x}{\sin x} = \lim_{x\to 0} \frac{1}{\cos x} = 1$

c) $\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sec^2 x}{1} = 1$

d) $\lim_{x\to 1} \frac{x-1}{x^2-1} = \lim_{x\to 1} \frac{1}{2x} = \frac{1}{2}$

e) $\lim_{x\to \infty} \frac{\ln x}{x} = \lim_{x\to \infty} \frac{\frac{1}{x}}{1} = \lim_{x\to \infty} \frac{1}{x} = 0$

f) $\lim_{x\to \infty} \frac{1+e^x}{xe^x} = \lim_{x\to \infty} \frac{e^x}{xe^x+e^x} = \lim_{x\to \infty} \frac{1}{x+1} = 0$

g) $\lim_{x\to \infty} \frac{\frac{1}{x^4}-\frac{1}{x^2}}{\ln x} = \lim_{x\to \infty} \frac{-\frac{4}{x^5}+\frac{2}{x^3}}{\frac{1}{x}} = \lim_{x\to \infty} \frac{-\frac{4}{x^4}+\frac{2}{x^2}}{=}0$

h) $\lim_{x\to \infty} \frac{\ln(x^2)}{(\ln x)^2} = \lim_{x\to \infty} \frac{\frac{1}{x^2}2x}{2(\ln x)\frac{1}{x}} = \lim_{x\to \infty} \frac{1}{\ln x} = 0$

Answers to: Forbidden questions

L'Hôpital's rule only applies to limits of the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$. This does not describe any of the limits in this section. Specifically:

a) $\lim_{x\to \pi} \frac{\sin x}{x} = \frac{0}{1} = 0$

b) $\lim_{x\to 0} \frac{\cos x}{x}$ has the form $\frac{1}{0}$

c) $\lim_{x\to -1} \frac{x-1}{x^2-1}$ has the form $\frac{-2}{0}$

d) $\lim_{x\to 4} \frac{x^2-16}{e^x} = \frac{0}{e^{16}} = 0$

e) $\lim_{x\to 4} \frac{x^2}{e^x} = \frac{16}{e^{16}}$

f) $\lim_{x\to \infty} \frac{e^{-x}}{x}$ has the form $\frac{0}{\infty}$

Problem Set 26: Extrema

Key skills: Derivatives, Critical points, Slopes

Practice Problems

Find the x values of the critical points of each of the following functions, and classify each as a local minimum, local maximum, or neither. As a bonus, determine if any of the critical points are *global* minima or maxima.

Check your answers by plotting the functions with Desmos (https://www.desmos.com) or other graphing software.

$a)\quad f(x) = (x-2)^3$ $\qquad b)\quad f(x) = xe^{-x}$ $\qquad c)\quad f(x) = x^2e^{-x}$ $\qquad d)\quad f(x) = xe^{-x^2}$

$e)\quad f(x) = \sqrt{x^2+4}$ $\qquad f)\quad f(x) = x\ln x$ $\qquad g)\quad f(x) = \dfrac{x^3}{3} - 4x + 4$

$h)\quad f(x) = \sin(x^2)$ $\qquad i)\quad f(x) = \sin^2 x$ $\qquad j)\quad f(x) = \dfrac{x^2+6x-1}{x^2+2}$

Answers

Remember that it's very useful to simplify and factor the derivative wherever possible (or at least convenient), since that makes the zeroes much easier to find. To determine whether an extremum is global or just local, remember to consider the limits as x goes to $\pm\infty$.

(a) $f'(x) = 3(x-2)^2 = 0 \implies x = 2$. $f'(x)$ is positive on both sides of $x = 2$ the function is increasing everywhere, so $x = 2$ is neither a minimum nor a maximum.

(b) $f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x} = 0 \implies x = 1$. $f'(x)$ is positive for $x < 1$, and negative for $x > 1$, so this is a (global) maximum.

(c) $f'(x) = 2xe^{-x} - x^2e^{-x} = (2x - x^2)e^{-x} = x(2-x)e^{-x} = 0 \implies x = 0$ or $x = 2$. $f'(x)$ is negative for $x < 0$, positive for $0 < x < 2$, and negative for $x > 2$, so $x = 0$ is a (global) minimum and $x = 2$ is a (local) maximum.

(d) $f'(x) = e^{-x^2} - 2x^2e^{-x^2} = (1-2x^2)e^{-x^2} = 0 \implies x = \pm\frac{1}{\sqrt{2}}$ (roughly ±0.707). $f'(x)$ is negative for $x < -\frac{1}{\sqrt{2}}$, positive for $\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$, and negative for $x > \frac{1}{\sqrt{2}}$, so $x = -\frac{1}{\sqrt{2}}$ is a (global) minimum and $x = \frac{1}{\sqrt{2}}$ is a (global) maximum.

(e) $f'(x) = \frac{x}{\sqrt{x^2+4}} = 0 \implies x = 0$. $f'(x)$ is negative for $x < 0$ and positive for $x > 0$, so $x = 0$ is a (global) minimum.

(f) $f'(x) = \ln x + 1 = 0 \implies x = e^{-1} = \frac{1}{e}$ (roughly 0.367). $f'(x)$ is negative for $x < \frac{1}{e}$ and positive for $x > \frac{1}{e}$, so $x = \frac{1}{e}$ is a (global) minimum.

(g) $f'(x) = x^2 - 4 = (x-2)(x+2) = 0 \implies x = \pm2$. $f'(x)$ is positive for $x < -2$, negative for $-2 < x < 2$, and positive for $x > 2$, so $x = -2$ is a (local) maximum and $x = 2$ is a (local) minimum.

(h) $f'(x) = 2x\cos(x^2) = 0 \implies x = 0$ or $x = \pm\sqrt{(n+\frac{1}{2})\pi}$, $n = 0, 1, 2, \ldots$. $f'(x)$ is negative just to the left of $x = 0$ and positive just to the right, so $x = 0$ is a (local) minimum. Moving out from here, $f'(x)$ flips sign after each critical point, so the critical points alternate between maxima and minima. All extrema are local.

(i) $f'(x) = 2\sin x\cos x = 0 \implies x = n\left(\frac{\pi}{2}\right)$, $n = 0, \pm1, \pm2, \ldots$. $f'(x)$ is negative just to the left of $x = 0$ and positive just to the right, so $x = 0$ is a minimum. Moving out from here, $f'(x)$ flips sign after each critical point, so the critical points alternate between maxima and minima. All extrema are local.

(j) $f'(x) = \frac{(2x+6)(x^2+2)-(x^2+6x-1)(2x)}{(x^2+2)^2} = \frac{-6x^2+6x+12}{(x^2+2)^2} = \frac{-6(x+1)(x-2)}{(x^2+2)^2} = 0 \implies x = -1$ or $x = 2$. $f'(x)$ is negative for $x < -1$, positive for $-1 < x < 2$, and negative for $x > 2$, so $x = -1$ is a (global) minimum, and $x = 2$ is a (global) maximum.

Problem Set 27: Optimization

Key skills: Optimization

Practice Problems

1. If there is a square prism with height, width h and length 1 m, determine the maximum volume of the prism if the surface area is 1 m^2.

2. The area of an ellipse with semi-axes a and b is πab. Find the maximum area under the constraint $a + b = 1$. Then find the maximum area under the constraint $a^2 + b^2 = 1$.

3. Two circles have centres which are exactly 1 m apart. The circles touch each other but do not overlap. What is the minimum of the sum of the areas of the two circles in this situation? *(Hint: The radii do not have to be the same!)*

Answers

1. Surface area is $2h^2 + 4hl = 1$. We solve for l and subsitute into the volume formula $V = h^2 l$. This gives

$$V = \frac{h - 2h^3}{4} \qquad V'(h) = \frac{1}{4} - \frac{6h^2}{4}$$

Solving for the critical points gives $h = \frac{1}{\sqrt{6}}$. We check that this is a maximum by seeing that the sign of the derivative changes from positive to negative. Then $l = \frac{1}{\sqrt{6}}$ as well, and we conclude that the cube has the maximum volume.

2. Process in both parts is the same as the previous question, using the constraint to remove a variable, and using the first derivative and critical points to find the maximum. The maximum area for the first part is $\pi/4$, and for the second part is $\pi/2$.

3. If r_1 and r_2 are the two radii, then the areas are πr_1^2 and πr_2^2 with the restriction that $r_1 + r_2 = 1$. Therefore $r_2 = 1 - r_1$. The total area is

$$A = \pi r_1^2 + \pi(1 - r_1)^2 = \pi(1 - 2r_1 + 2r_1^2)$$

If we take the derivative and optimize, we find a local minimum at $r_1 = \frac{1}{2}$. Therefore, the total area is minimized when both radii are $1/2$ and the minimum sum of the areas of both circles is $\pi/2$ units squared.

Problem Set 28: Marginal Analysis

Part of the larger field of "Cost-Benefit Analysis".

Key skills: Optimization, Intersections

Practice Problems

For each of the following pairs of cost and benefit functions, find the point of maximum net benefit, and the range(s) of production rates with a positive net benefit. Determine whether production should be increased, decreased, or neither at production levels $x = 1, 2, 3$. (Recall that x is the production rate, $C(x)$ is the cost of producing at rate x, and $B(x)$ is the benefit (income from sale) obtained from production at rate x.)

In most of these cases you'll need to have a computer calculate the final solutions. Your main job is to come up with the equations for the computer to solve. One way is to ask Wolfram Alpha to "solve" the function (for example: `solve 0=x^3-8x^2+4x-25`). You may need to click "Approximate forms" to get a numerical answer.

a) $C(x) = x^2 + 6$
$B(x) = -x^2 + 8x$

b) $C(x) = \dfrac{x^2}{8}$
$B(x) = \sqrt{x}$

c) $C(x) = \dfrac{x^2 + 1}{4}$
$B(x) = \begin{cases} \frac{x^2}{2} & x \leq 2 \\ 2x - 2 & x > 2 \end{cases}$

d) $C(x) = e^{x/5}$
$B(x) = 2\ln(x + 1)$

e) $C(x) = x + 4$
$B(x) = \dfrac{10x}{x + 1}$

Answers

Define $N(x) = B(x) - C(x)$ to be the net benefit.

$$a) \quad N(x) = -2x^2 + 8x - 6 = -2(x-1)(x-3)$$

Net benefit is positive for $1 < x < 3$.

$$N'(x) = -4x + 8$$

The derivative is 0 at $x = 2$, so since $N(x)$ is concave-down (negative x^2 term) this is the point of maximum net benefit. $N'(x)$ is positive at $x = 1$, zero at $x = 2$, and negative at $x = 3$, so production should increase, stay the same, and decrease at these levels, respectively.

$$b) \quad N(x) = \sqrt{x} - \frac{x^2}{8}$$

Net benefit is positive for $0 < x < 4$.

$$N'(x) = \frac{1}{2\sqrt{x}} - \frac{x}{4}$$

The derivative is 0 at $x = 2^{2/3} \approx 1.587$, so since $N(x)$ is concave-down this is the point of maximum net benefit. $N'(x)$ is positive at $x = 1$, so production should increase from there; it is negative at $x = 2$ and 3, so production should be decreased from those values.

$$c) \quad N(x) = \begin{cases} \frac{x^2}{2} - \frac{x^2+1}{4} & x \le 2 \\ 2x - 2 - \frac{x^2+1}{4} & x > 2 \end{cases}$$

or

$$N(x) = \begin{cases} \frac{3x^2-1}{4} & x \le 2 \\ \frac{-x^2+8x-9}{4} & x > 2 \end{cases}$$

Net benefit is positive for $\frac{1}{\sqrt{3}} < x < 4 + \sqrt{7}$ (approximately $0.577 < x < 6.646$).

$$N'(x) = \begin{cases} \frac{3x}{2} & x \le 2 \\ -\frac{x}{2} + 2 & x > 2 \end{cases}$$

This derivative is 0 at $x = 0$ and $x = 4$; only $x = 4$ has positive net benefit, so this must be the maximum. $N'(x)$ is positive at $x = 1, 2, 3$ so production should be increased in each of those cases.

d) $N(x) = 2\ln(x+1) - e^{x/5}$

Net benefit is positive for approximately $(0.798 < x < 7.179)$.

$$N'(x) = \frac{2}{x+1} - \frac{1}{5}e^{x/5}$$

This derivative is 0 at $x \approx 3.736$, so that production gives maximum net benefit. $N'(x)$ is positive at $x = 1, 2, 3$ so production should be increased in each of those cases.

e) $N(x) = \dfrac{10x}{x+1} - x - 4 = \dfrac{10x - (x^2 + 5x + 4)}{x+1} = \dfrac{-x^2 + 5x - 4}{x+1} = \dfrac{-(x-1)(x-4)}{x+1}$

Net benefit is positive for $1 < x < 4$.

$$N'(x) = \frac{(-2x+5)(x+1) - (-x^2+5x-4)(1)}{(x+1)^2} = \frac{-x^2 - 2x + 9}{(x+1)^2}$$

This derivative is 0 at $x = \sqrt{10} - 1 \approx 2.162$, so since $N(x)$ is concave-down this is the point of maximum net benefit. $N'(x)$ is positive at $x = 1$ and 2, so production should increase from there. $N'(x)$ is negative at $x = 3$ so production from there should be decreased.

Problem Set 29: Curve Sketching

Key skills: Curve Sketching

Practice Problems

Sketch the following functions, using all the function properties discussed in class.

a) $f(x) = \dfrac{x^2}{1+x^2}$ *b*) $f(x) = \ln(x^2+4)$

c) $f(x) = e^{-x}\sin(x+\pi)$ *d*) $f(x) = x^2 + \dfrac{1}{x^3}$

Answers

a) The domain is $\mathbb{R}$. The range is $[0,1)$. The function has even symmetry. There is an x and y-intercept at $(0,0)$ and no other intercepts. The function is always positive. The limit as $x \to \pm\infty$ is 1, so $y = 1$ is a horizontal asymptote in both the positive and negative directions.

The first derivative is

$$f'(x) = \frac{2(1+x^2) - x^2(2x)}{(1+x^2)^2} = \frac{2x}{(1+x^2)^2)}$$

This has a root at $x = 0$. The function is decreasing on $(-\infty, 0)$ and increasing on $(0,\infty)$. $x = 0$ is a local minimum.

The second derivative is

$$f''(x) = \frac{2(1+x^2)^2 - 2x(2x)(2)(1+x^2)}{(1+x^2)^4} = \frac{2(1+x^2) - 8x^2}{(1+x^2)^3} = \frac{2-6x^2}{(1+x^2)^3}$$

This has roots at $x = \pm\sqrt{\frac{1}{3}}$. The function is concave down on $\left(-\infty, -\sqrt{\frac{1}{3}}\right)$, concave up on $\left(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right)$ and concave down on $\left(\sqrt{\frac{1}{3}}, \infty\right)$. Both roots are inflection points.

b) The domain is $\mathbb{R}$. The range is $[\ln 4, \infty)$. The function has even symmetry. There is a y-intercept at $(0, \ln 4)$. There are no x intercepts. The function is always positive. The limit as $x \to \pm\infty$ is ∞, so there are no asymptotes.

The first derivative is

$$f' = \frac{2x}{x^2+4}$$

This has a root at $x = 0$. The function is decreasing on $(-\infty, 0)$ and increasing on $(0,\infty)$. $x = 0$ is a local minimum.

The second derivative is

$$f''(x) = \frac{2(x^2+4) - 2x(2x)}{(x^2+4)^2} = \frac{8-2x^2}{(x^2+4)^2}$$

This has a root at $x = \pm 2$. The function is concave down on $(-\infty, -2)$, concave up on $(-2, 2)$ and concave down on $(2,\infty)$. Both roots are inflection points.

c) The domain is $\mathbb{R}$ and the range is $\mathbb{R}$. There is no symmetry. There is a y-intercept at $(0,0)$ and infinitely many x-intercepts at all multiples of π. The limit as $x \to \infty$ is 0, so $y = 0$ is a horizontal asymptote in the positive direction. The limit as $x \to -\infty$ does not exist.

The first derivative is

$$f'(x) = -e^{-x}\sin(x+\pi) + e^{-x}\cos(x+\pi) = e^{-x}(\cos(x+\pi) - \sin(x+\pi)$$

This has infinitely many roots, whenever sin and cos are equal. (Those values are slightly tricky to calculate). All of these critical points are maxima or minima, as the sinusoidal part of the functions oscillates up and down.

The second derivative is

$$f''(x) = e^{-x}\sin(x+\pi) - 2e^{-x}\cos(x+\pi) - e^{-x}\sin(x+\pi) = -2e^{-x}\cos(x+\pi)$$

There are infinitely many roots, at all odd multiples of $\pi/2$. All these roots are inflection points, as the sinusoidal part of the function switches between concave up to concave down.

d) The domain is all $x \neq 0$. The range is $\mathbb{R}$. There are no y-intercepts. There are no x-intercepts. There is no symmetry. The limits as $x \to 0$ are $\pm\infty$, so there is a vertical asymptote at $x = 0$. The limits as $x \to \pm\infty$ are ∞ so there are no horizontal asymptotes.

The first derivative is

$$f'(x) = 2x - \frac{1}{3x^4}$$

It has a root at $x = \sqrt[5]{\frac{1}{6}}$. The function is decreasing on $(-\infty, 0)$, decreasing on $\left(0, \sqrt[5]{\frac{1}{6}}\right)$ and increasing on $\left(\sqrt[5]{\frac{1}{6}}, \infty\right)$. $x = \sqrt[5]{\frac{1}{6}}$ is a local minimum.

The second derivative is

$$f'' = 2 + \frac{1}{12x^5}$$

It has a root at $x = \sqrt[5]{\frac{-1}{24}}$. The function is concave up on $\left(-\infty, \sqrt[5]{\frac{-1}{24}}\right)$. The function is concave down on $\left(\sqrt[5]{\frac{-1}{24}}, 0\right)$. The function is concave up on $(0, \infty)$. $x = \sqrt[5]{\frac{-1}{24}}$ is an inflection point.